JEE ASPIRANTS

# MANZIL

**Mathematics**

## Continuity, Differentiability & MOD

**One Shot**

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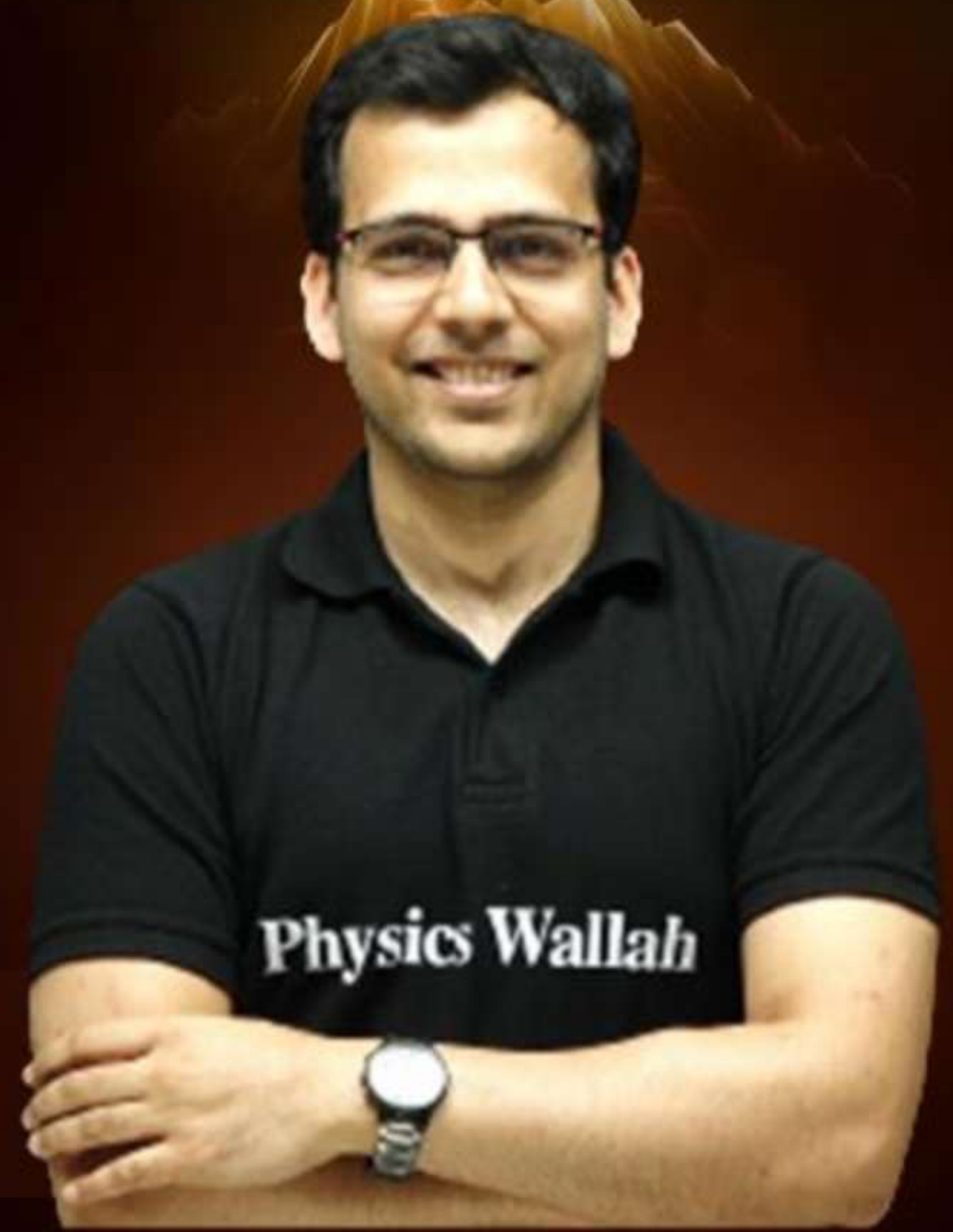


# Today's Targets

1. Continuity at a point $x = a$
2. Types of Discontinuity
3. Continuity in an Interval
4. Properties of Continuous Functions
5. Differentiability at a point $x = a$
6. Differentiability in an Interval
7. Properties of differentiable Functions
8. Methods of Differentiation ( M.O.D.)
9. PYQs – 2024 Jan Attempt

# Continuity at a point x = a

Ex

$$f(x) = \begin{cases} 2x^2+3 & \text{if } x>1 \\ 3x+2 & \text{if } x<1 \\ 3 & \text{if } x=1 \end{cases}$$

$\lim_{x\to a} f(x)$

$LHL = \lim_{x\to a^-} f(x)$

$RHL = \lim_{x\to a^+} f(x)$

RHL: $\lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} (2x^2+3) = 2(1)^2+3 = 5$

For limit to exist LHL = RHL = finite

LHL: $\lim_{x\to 1^-} (3x+2) = 5$

For continuity at $x = a$

$$\boxed{LHL = RHL = f(a) = \text{Finite}}$$

$f(1) = 3$

## Continuity at a point x = a

A function f(x) is said to be continuous at $x = a$,
if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$ = Finite

i.e. L. H. L. = R. H. L. = value of the function at 'a' i.e., $\lim_{x \to a} f(x) = f(a)$.

If f(x) is not continuous at $x = a$, we say that f(x) is discontinuous at $x = a$.

QUESTION [JEE Main 2021]

[Ans. 14]

$-a = 4/b$

$\boxed{ab = -4}$

Let $a, b \in R$, $b \neq 0$, define a function $f(x)=\begin{cases} a\sin\frac{\pi}{2}(x-1), & \text{for } x \le 0 \\ \frac{\tan 2x - \sin 2x}{bx^3}, & \text{for } x > 0 \end{cases}$.

If f is continuous at x = 0, then 10 − ab is equal to :

$-4$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$$

$$\text{RHL}: \lim_{x\to 0^+}\left(\frac{\tan 2x - \sin 2x}{bx^3}\right)$$

$$\frac{\frac{\sin 2x}{\cos 2x} - \sin 2x}{bx^3} = \frac{\sin 2x\left[\frac{1}{\cos 2x} - 1\right]}{bx^3} = \frac{\tan 2x\,(1-\cos 2x)}{bx^3} = \frac{\tan 2x \cdot 2\sin^2 x}{bx^3}$$

$$= \frac{\tan 2x \cdot 2}{bx} = \frac{2\tan 2x \cdot 2}{b \cdot 2x} = 4/b$$

LHL = f(0)

$$\lim_{x\to 0}\left(a\sin\frac{\pi}{2}(x-1)\right)$$

$$= a\sin\left(\frac{\pi}{2}(-1)\right) = -a\sin\frac{\pi}{2}$$

$$= \boxed{-a}$$

## QUESTION [JEE Main 2021 (Aug.)]

[Ans. D]

If the function $f(x) = \begin{cases} \frac{1}{x}\log_e\left(\frac{1+\frac{x}{a}}{1-\frac{x}{b}}\right), & \boxed{x < 0} \\ k, & \boxed{x = 0} \\ \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1}-1}, & \boxed{x > 0} \end{cases}$

is continuous at x = 0, then $\frac{1}{a} + \frac{1}{b} + \frac{4}{k}$ is equal to

(A) 4

(B) 5

(C) –4

(D) –5

$-4 - 1$

$= -5$

RHL: $f(0^+) = \lim_{x\to 0^+} \frac{\cos 2x - 1}{(\sqrt{x^2+1}-1)} \cdot \frac{\sqrt{x^2+1}+1}{(\sqrt{x^2+1}+1)}$

$\lim_{x\to 0^+} \frac{-2\sin^2 x(\sqrt{x^2+1}+1)}{x^2 + \cancel{1} - \cancel{1}}$

$= -2(\sqrt{x^2+1}+1)$

$-2(1+1) = -4$

$$\lim_{x\to 0^-} \frac{\ln(1+x/a)}{a\cdot \frac{x}{a}} + \frac{\ln(1-x/b)}{b\cdot\left(-\frac{x}{b}\right)}$$

1    1

$$LHL = \frac{1}{a} + \frac{1}{b} = \underbrace{-4 = K}$$

$$\boxed{\frac{4}{K} = -1}$$

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

$$+ \frac{\ln(1-x/b)}{-\frac{x}{b}\cdot(+b)}$$

1

$$\frac{1}{b}$$

$S = \{(-4, 2)\}$

**QUESTION [JEE Main 2024 (Jan. I)]**

[Ans.

Consider the function, $f(x) = \begin{cases} \dfrac{a(7x-12-x^2)}{b|x^2-7x+12|}, & x < 3 \\ 2^{\frac{\sin(x-3)}{x-[x]}}, & x > 3 \\ b, & x = 3, \end{cases}$ where $[x]$ denotes the greatest integer less than or equal to $x$. If $S$ denotes the set all ordered pairs $(a, b)$ such that $f(x)$ is continuous at $x = 3$, then the number of elements in $S$ is:

$(-4, 2)$

LHL: $-a/b$

(A) Infinitely many

(B) 4

(C) 1 ✓

(D) 2

RHL: $\lim_{x \to 3^+} 2^{\frac{\sin(x-3)}{x-[x]}}$

$\lim_{x \to 3^+} 2^{\frac{\sin(x-3)}{(x-3)}}$

$x - 3 = \theta$

$\lim_{\theta \to 0} 2^{\frac{\sin\theta}{\theta}}$, $\frac{\sin\theta}{\theta} \to 1$

$= 2^1 = 2$

$f(3) = b$

$2 = -\frac{a}{b} = b$

$\boxed{b = 2}$

$-\frac{a}{b} = b$

$-a = b^2$

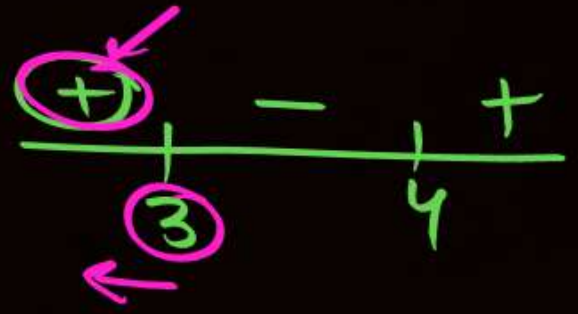
$-a = 4$

$\boxed{a = -4}$

LHL $\dfrac{a(7x-12-x^2)}{b\,|x^2-7x+12|} = -\dfrac{a}{b}\dfrac{(x^2-7x+12)}{|x^2-7x+12|}$

$$-\frac{a}{b}\,\frac{(x-3)(x-4)}{|(x-3)(x-4)|}$$

mod opens with +ve
when $x \to 3^-$




$$-\frac{a}{b}$$

QUESTION [JEE Main 2023]

[Ans. D]

If the function

$$f(x)=\begin{cases}(1+|\cos x|)^{\frac{\lambda}{|\cos x|}}, & 0<x<\frac{\pi}{2}\\ \boxed{\mu,} & \boxed{x=\frac{\pi}{2}}\\ e^{\frac{\cot 6x}{\cot 4x}}, & \frac{\pi}{2}<x<\pi\end{cases}$$

$\boxed{\text{is continuous at } x=\frac{\pi}{2}}$, then $9\lambda+6\log_e\mu+\mu^6-e^{6\lambda}$ is equal to

(A) 11

(B) 8

(C) $2e^4+8$

(D) 10

$6+6\times 2/3+\cancel{e^4}-\cancel{e^4}$

$6+4$

$=\boxed{10}$

LHL $\lim_{x\to\pi/2^-}(1+|\cos x|)^{\frac{\lambda}{|\cos x|}}$

$\lim_{x\to\pi/2^-}(1+\cos x)^{\lambda/\cos x}$

$\boxed{(1)^\infty}$

LHL $= e^{\cancel{\cos x}\cdot\lambda/\cancel{\cos x}} = e^\lambda$

RHL : $\lim_{x\to\pi/2^+} e^{\frac{\cot 6x}{\cot 4x}}$

$=\lim_{x\to\pi/2^+} e^{\frac{\tan 4x}{\tan 6x}}$

let $\boxed{x-\pi/2=\theta} \Rightarrow x=\pi/2+\theta$

$$\cancel{\lim_{x \to \pi/2^+} e^{\frac{\tan 4x}{\frac{\tan 6x}{6x} \cdot 6x}}}$$

$$\lim_{\theta \to 0} e^{\frac{\tan 4(\pi/2+\theta)}{\tan 6(\pi/2+\theta)}} = e^{\frac{\tan(2\pi+4\theta)}{\tan(3\pi+6\theta)}} = e^{\frac{\tan 4\theta}{\tan 6\theta}}$$

$$= e^{\frac{\frac{\tan 4\theta}{4\theta} \cdot 4\theta}{\frac{\tan 6\theta}{6\theta} \cdot 6\theta}}$$

$$e^{\frac{4\theta}{6\theta}} = \boxed{e^{2/3}}$$

$$\mu = e^{2/3} = e^{\lambda}$$

$$\boxed{\lambda = 2/3} \Rightarrow 9\lambda = 6$$

$$6\lambda = 4$$

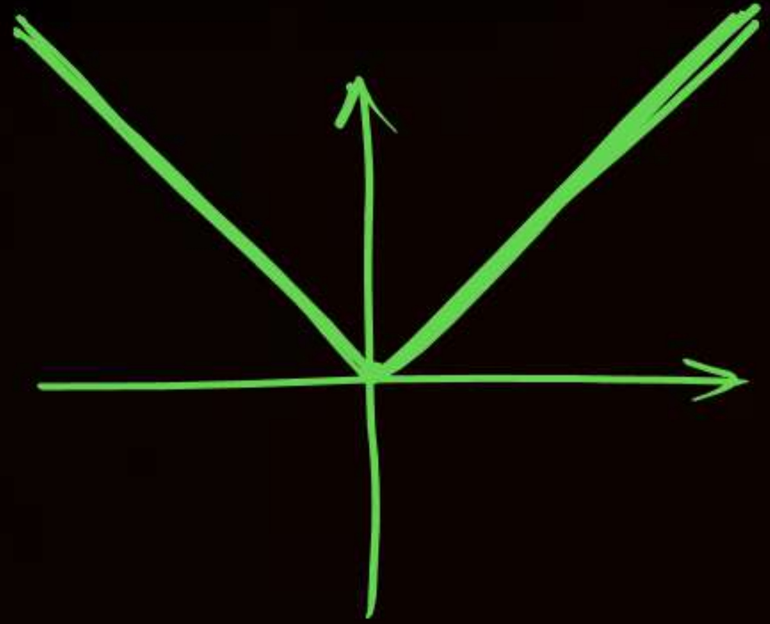
$$e^{6\lambda} = e^4$$

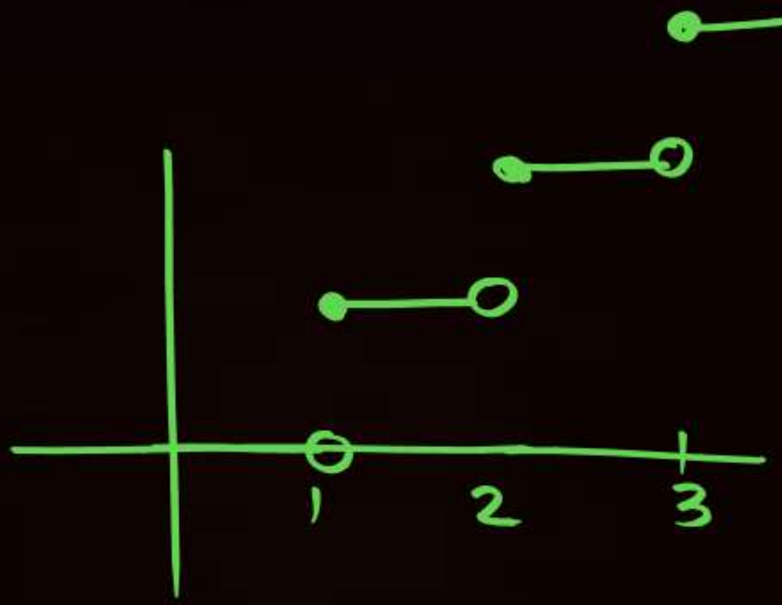
$$\mu = e^{2/3} \Rightarrow \mu^6 = e^4$$

$$\Rightarrow \ln \mu = 2/3$$

# Geometrical Interpretation of Continuity at a point x = a

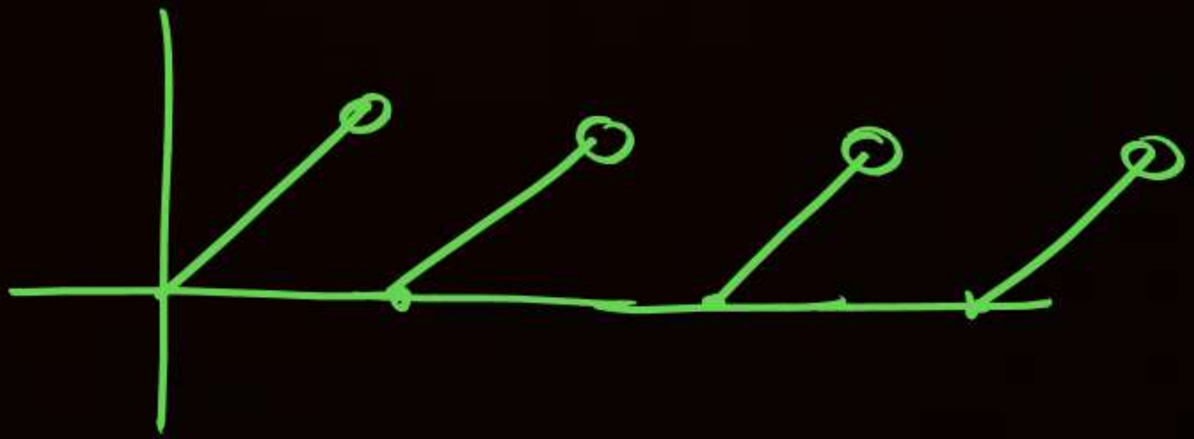
$|x|$

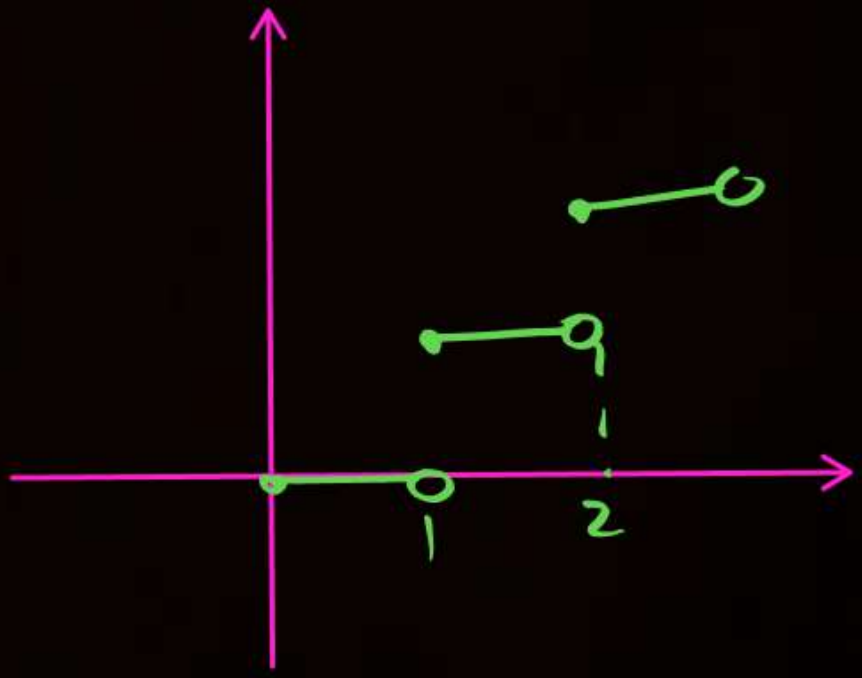


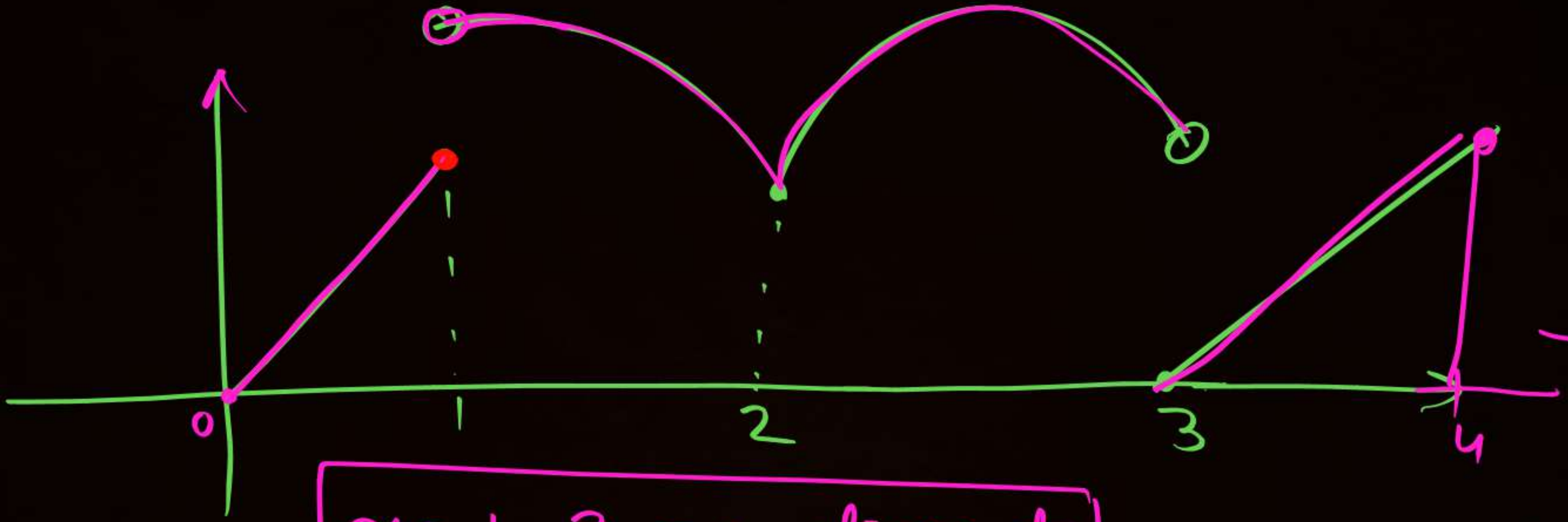


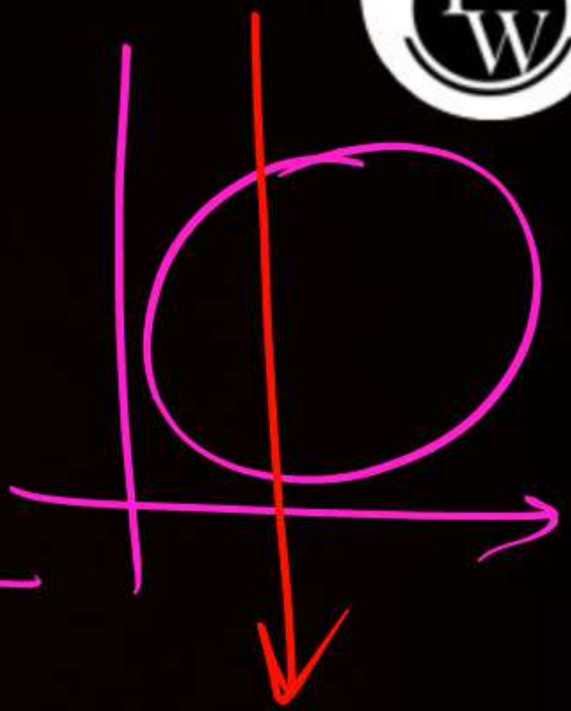

$$\lim_{x\to 2} [x] \begin{cases} \nearrow RHL = 2 \\ \searrow LHL = 1 \end{cases}$$

$y = \{x\}$

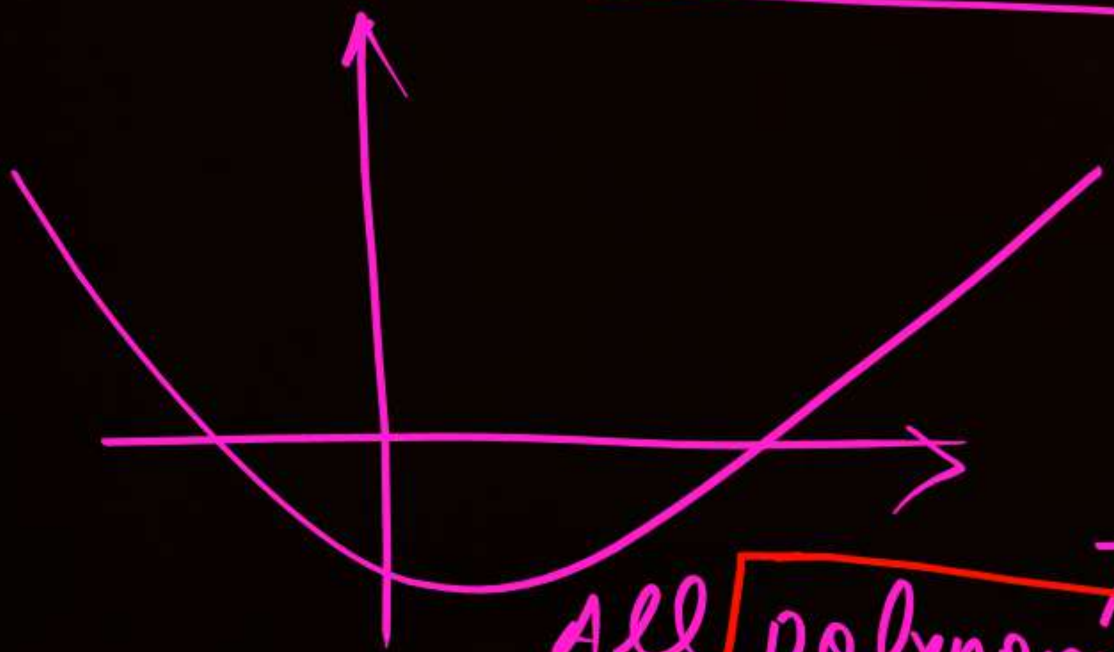




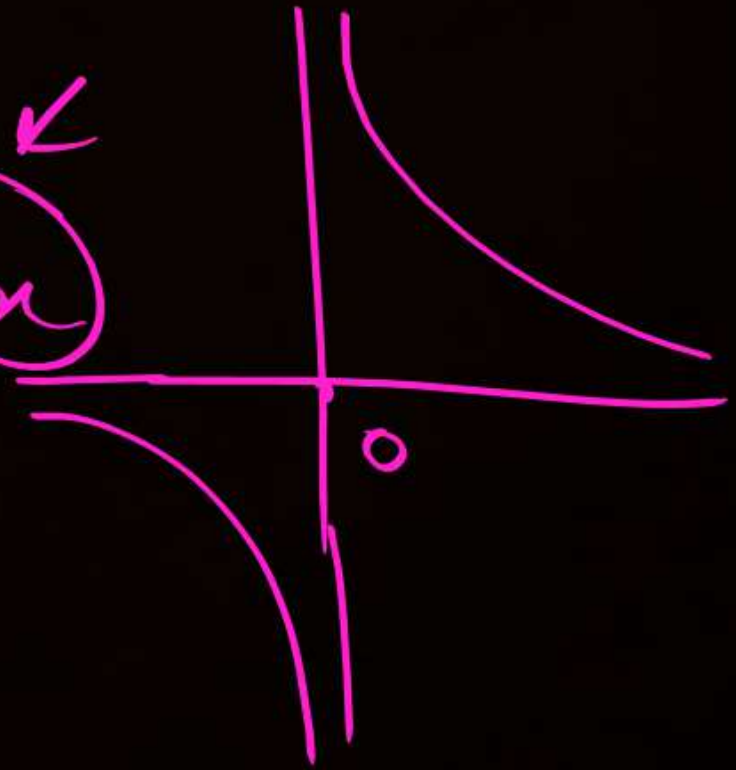


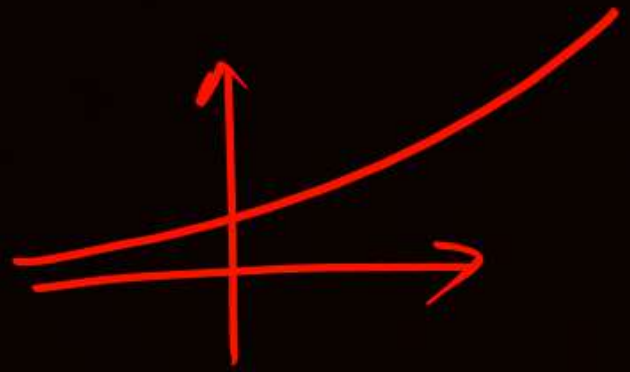
$\boxed{x = 1, 3 \rightarrow \text{discont}}$



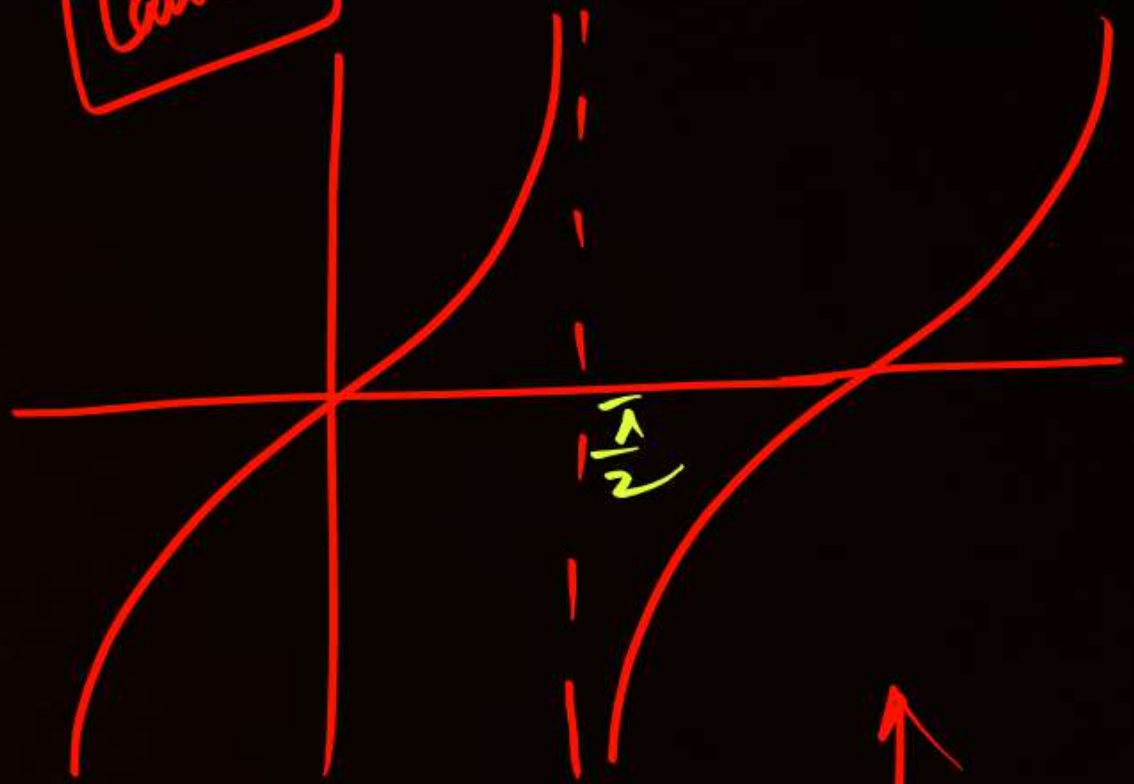
$y = 1/x$




All $\boxed{\text{polynomial}}$ fns are always cont

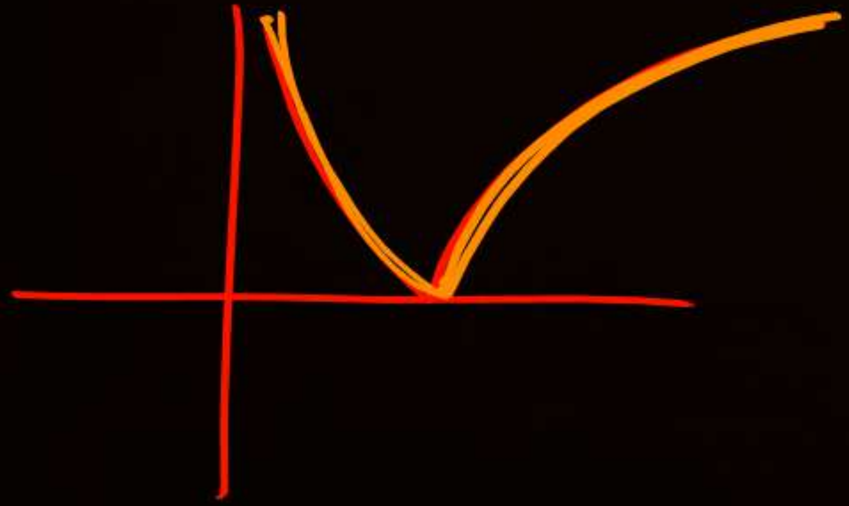


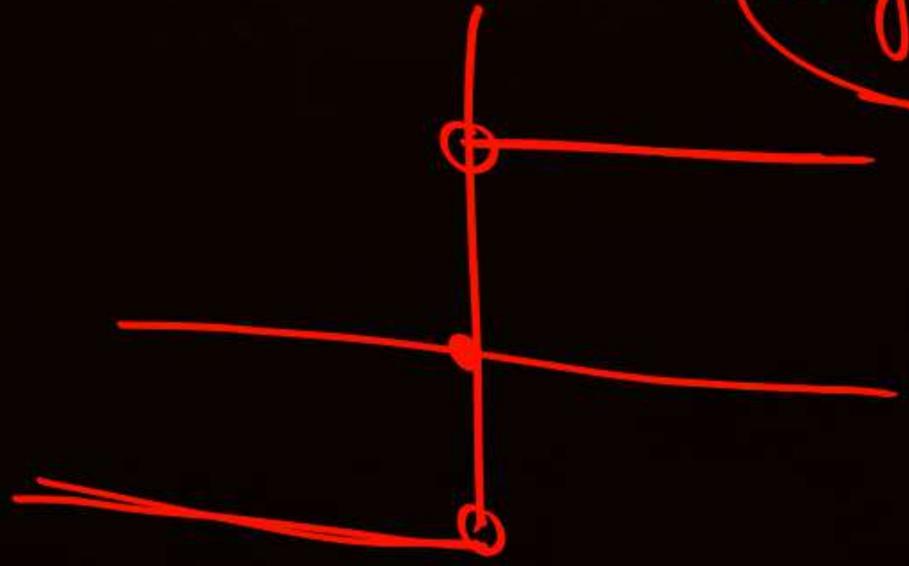
$\boxed{\tan x}$

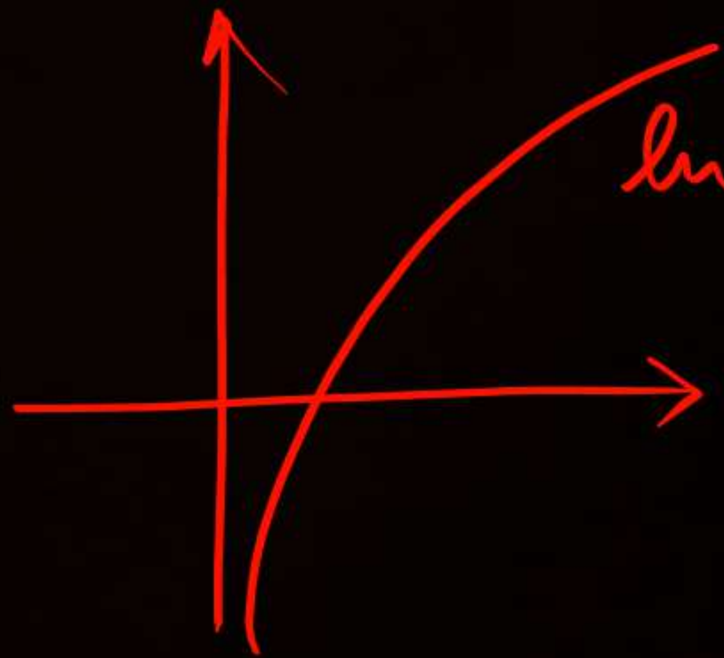



$y = |\ln x|$

Sgn(x)







$\ln x$

# Types of Discontinuity

$\underbrace{LHL = RHL}$ = $\underbrace{f(a)}$ = Finite

Removable discont

1) when $\boxed{LHL = RHL} \neq \underbrace{f(a)}$

a) Missing →

b)

# Removable Discontinuity

Ex →

$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ \cancel{5} & \text{if } x = 2 \end{cases}$ Isolated

LHL = RHL → Finite

If, $\lim\limits_{x \to a} f(x)$ exists but is not equal to f(a) then the function is said to have a removable discontinuity.

(i) Missing Point Discontinuity:

Where $\lim\limits_{x \to a} f(x)$ exists finitely but f(a) is not defined.

$f(x) = \dfrac{x^2-4}{x-2}$ at $x = 2$ it has missing pt discontly

$\lim\limits_{x \to 2} \left(\dfrac{x^2-4}{x-2}\right) = \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = 4$

but f(2) is n.d.

(ii) Isolated Point Discontinuity:

Where $\lim\limits_{x \to a} f(x)$ exists and f(a) also exists but;

Removable → $\boxed{\text{Missing pt}}$

It is possible to define $\boxed{f(a) = LHL = RHL}$

Ex → $f(x) = \dfrac{x^2-4}{x-2}$

✓ $$\boxed{f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}}$$

$\boxed{\text{Isolated pt} \rightarrow}$

"Re-define" $f(a) = LHL = RHL$

$$f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$$

# Ir-removable Discontinuity

If $\boxed{\lim_{x \to a} f(x) \text{ doesn't exist}}$ then the function is said to have a removable discontinuity at point x = a.

Irremovable discontinuity can be further classified as:

Jump of discont = |LHL − RHL| =

(i) Finite type irremovable discontinuity LHL & RHL both are finite but unequal

$f(x) = [x]$ at $\boxed{x=3}$ RHL = 3, LHL = 2

(ii) Infinite type irremovable discontinuity atleast one of LHL or RHL → ∞

$$f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(iii) Oscillatory type irremovable discontinuity LHL or RHL or both oscillate

Ex → $f(x) = \begin{cases} \sin(1/x) , & x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

has oscillatory discont at $x = 0$

$\lim_{x \to 0} \sin(1/x)$

$1/x \to \infty$ $\quad \sin(\pm\infty)$

$[-1, 1]$

Discont

Removable (LHL=RHL)
- → missing
- → isolated

Irremovable
- →
- →
- →

**QUESTION**

Let $f(x) = \begin{cases} \dfrac{\ln(\cos x)}{\sqrt[4]{1+x^2}-1} & \text{if } x > 0 \\ \dfrac{e^{\sin 4x}-1}{\ln(1+\tan 2x)} & \text{if } x < 0 \end{cases}$

Is it possible to define f(0) to make the function continuous at x = 0.

If yes what is the value of f(0), if not then indicate the nature of discontinuity.

$$LHL = \lim_{x\to 0^-} \frac{\dfrac{e^{\sin 4x}-1}{\sin 4x}\cdot \sin 4x}{\dfrac{\ln(1+\tan 2x)}{\tan 2x}\cdot \tan 2x}$$

$$\lim_{x\to 0^-} \frac{\dfrac{\sin 4x}{4x}\cdot 4x}{\dfrac{\tan 2x}{2x}\cdot 2x} = 2$$

No

$$RHL: \lim_{x\to 0^+} \frac{\ln(\cos x)}{(1+x^2)^{1/4}-1} = \frac{\frac{1}{\cos x}(-\sin x)}{\frac{1}{4}(1+x^2)^{-3/4}(2x)} = \frac{-2\tan x}{(1+x^2)^{-3/4}\,x}$$

$$= -\frac{2}{(1+x^2)^{-3/4}} = \boxed{-2}$$

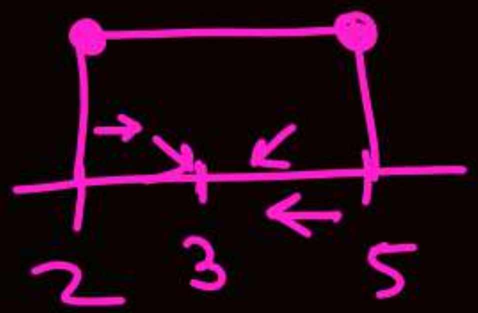
"Finite type irremovable discont."

# Continuity in an interval

↳ If $f$ is cont in $x \in (2, 5) \Rightarrow f(x)$ is cont at all pts b/w 2 & 5.

↳ If $f$ is cont in $x \in [2, 5] \Rightarrow$

check one sided limit at boundary pt




at $x = 2$

$$\Rightarrow \lim_{x \to 2^+} f(x) = f(2)$$

at $x = 5$

$$\lim_{x \to 5^-} f(x) = f(5)$$

# Continuity on an interval

(1) A function $f(x)$ is said to be continuous in an open interval $(a, b)$ if it is continuous at each and every point of $(a, b)$ i.e., $y = [x]$ is continuous in $(1, 2)$, where [ ] is greatest integer function.

(2) A function $f(x)$ is said to be continuous in a closed interval $[a, b]$ if

(a) it is continuous in $(a, b)$

(b) value of the function at "$b$" is equal to left hand limit at "$b$" i.e.,
$f(b) = \lim_{x \to b^-} f(x)$

(c) value of the function at "$a$" is equal to right hand limit at "a" i.e.,
$f(a) = \lim_{x \to a^+} f(x)$

## QUESTION [JEE Main 2019 (April)]

[Ans. D]

Let $f : [-1, 3] \to R$ be defined as $f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + [x], & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3 \end{cases}$ where [t] denotes g.i.f. Then f is discontinuous at :

(A) four or more points

(B) only one point

(C) only two points

(D) only three points

$$f(x) = \begin{cases} x + [x] \leftarrow 0, & x \in [0, 1) \\ -x + [x] \leftarrow -1, & x \in (-1, 0) \\ \boxed{x + x}, & 1 \le x < 2 \\ x + 2, & 2 \le x < 3 \\ 6 & x = 3 \end{cases}$$

$$f(x) = \begin{cases} x, & x \in [0, 1) \\ -x-1, & x \in [-1, 0) \\ 2x, & 1 \le x < 2 \\ x+2 & 2 \le x < 3 \end{cases}$$

6 if $x=3$

$$f(x) = \begin{cases} x & x \in [0,1) \\ -x-1 & x \in [-1,0) \\ 2x, & 1 \le x < 2 \\ x+2, & 2 \le x < 3 \\ 6, & x = 3 \end{cases}$$

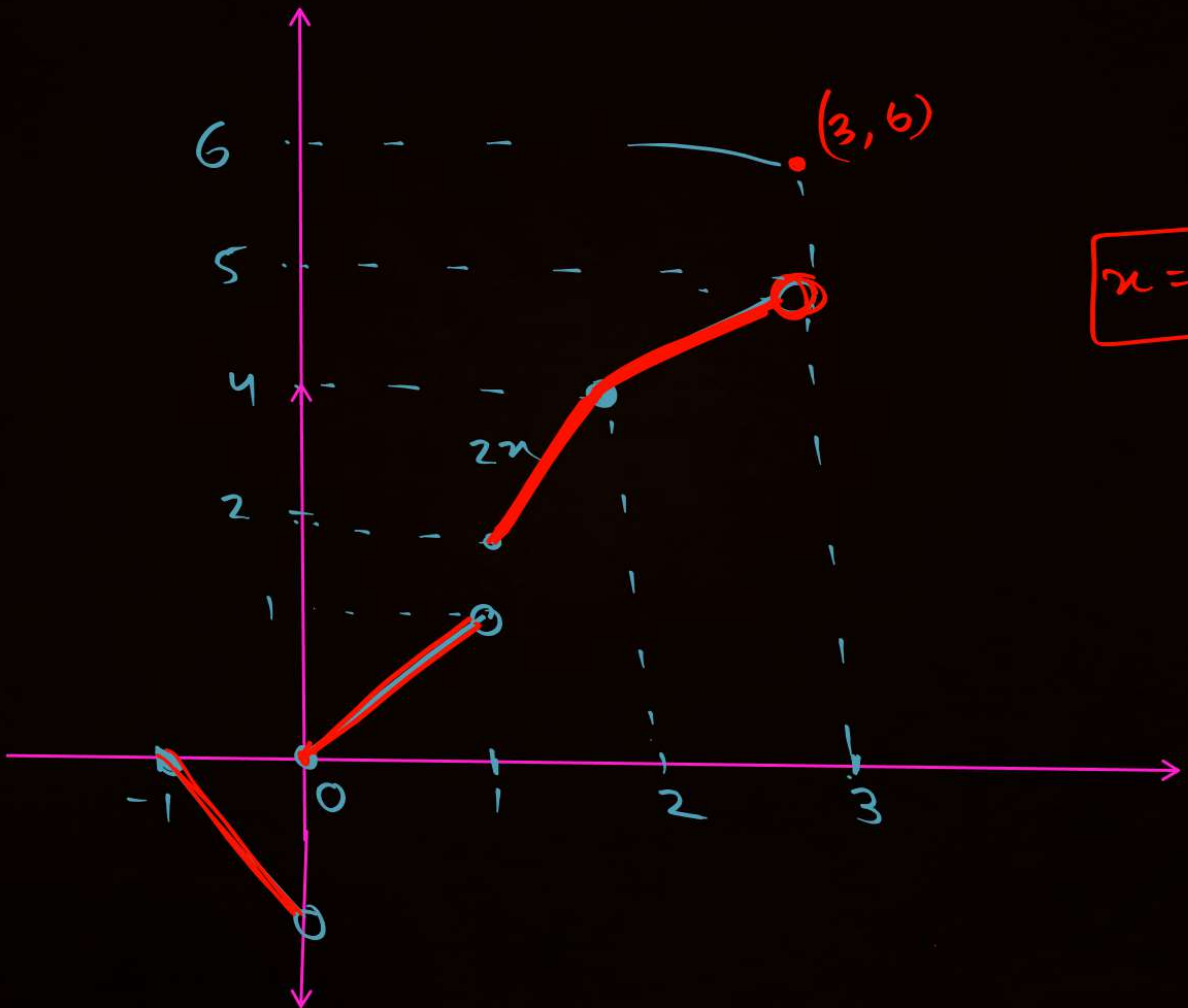
1) at $x = 0$ $\begin{cases} \nearrow RHL = 0 \\ \searrow LHL = -1 \end{cases} \Rightarrow$ dis cont

2) at $x = 1$ $\begin{cases} \nearrow RHL = 2(1) = 2 \\ \searrow LHL = 1 \end{cases} \rightarrow$ discont

3) at $x = 2$ $\begin{cases} \nearrow RHL = 2+2 = 4 \\ \searrow LHL = 4 \end{cases}$ $f(2) = 4 \longrightarrow$ cont

at $\boxed{x = 3}$

$\left.\begin{array}{l} f(3) = 6 \\ LHL = 5 \end{array}\right] \rightarrow$ dis cont




$\boxed{x=0,1,3}$

QUESTION [JEE MAIN 2023 (Apr.-I)]

[Ans. 2]

Let [x] be the greatest integer $\leq$ x. Then the number of points in the interval $(-2,1)$ where the function $f(x) = |[x]| + \sqrt{x - [x]}$ is discontinuous, is______.

$x \in [0,1) \Rightarrow f(x) = |0| + \sqrt{x-0} = \sqrt{x}$

$x \in [-1,0) \Rightarrow f(x) = 1 + \sqrt{x+1}$

$x \in (-2,-1) \Rightarrow f(x) = 2 + \sqrt{x+2}$

at $x = 0$ $\nearrow$ RHL $= 0$, $\searrow$ LHL $= 2$

at $x = -1$ $\nearrow$ RHL $= 1$, $\searrow$ LHL $= 3$

## QUESTION [JEE Main-2023 (April)]

$[x+a] = [x]+a$

[Ans. 25]

#

Let $f(x) = [a + 13\sin x]$ for $\boxed{0 < x < \pi}$, a is integer, where [t] denotes the greatest integer function. Then the number of points of discontinuity of f(x) is equal to ............... .

$$f(x) = a + [13\sin x]$$

$\sin x \in (0, 1]$

$13\sin x \in (0, 13]$

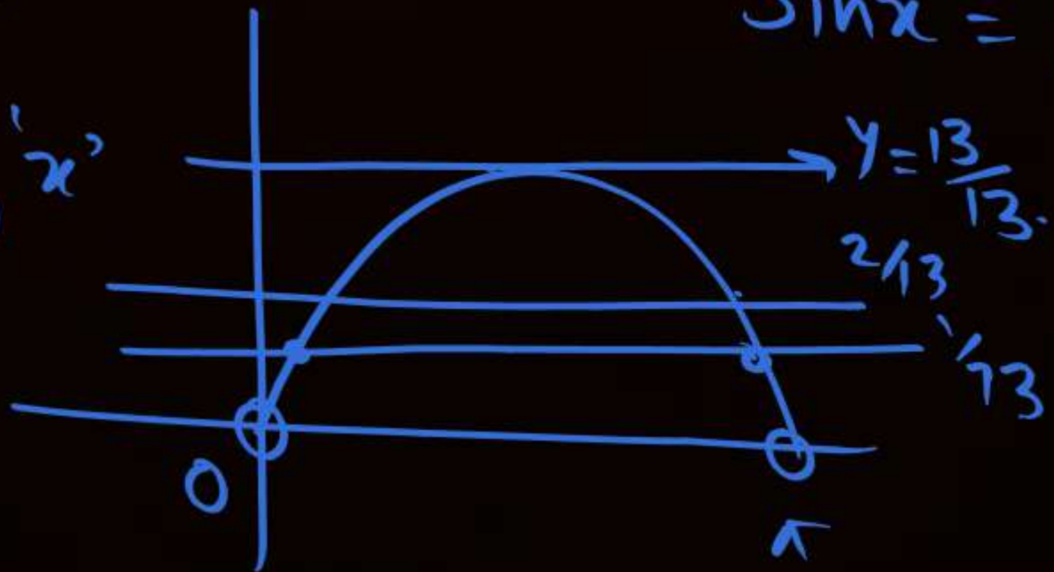
$13\sin x = 1, 2, 3, 4, 5, 6, \ldots 13$

$\sin x = \frac{1}{13}, \frac{2}{13}, \frac{3}{13}, \ldots \frac{13}{13}$

$12 \times 2 + 1$

$\sin x = \frac{1}{13}$

2 sol$^n$ of 'x'



# Properties of Continuous Functions

$\checkmark \rightarrow$ cont
$\times \rightarrow$ discont

| $f$ | $g$ | $f \pm g$ | $fg$ or $\frac{f}{g}$ |
|---|---|---|---|
| $\checkmark$ | $\checkmark$ | $\checkmark$ | $\checkmark$ |
| $\checkmark$ | $\times$ | $\times$ | ? |
| $\times$ | $\times$ | ? | ? |

Let $f(x)$ and $g(x)$ are continuous functions at $\boxed{x = a.}$
Then, $c\,f(x)$, $\underline{f(x) \pm g(x)}$, $f(x) \cdot g(x)$, $f(x)/g(x)$ $(g(a) \neq 0)$ are continuous at $x = a$ where $c$ is any constant.

Ex $\rightarrow$ $f(x) = |x|$ at $x=0$
$g(x) = \cos x$ at $x=0$

If $f(x)$ is continuous & $g(x)$ is discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ will not necessarily be discontinuous at $x = a$.

If $f(x)$ and $g(x)$ both are discontinuous at $x = a$ then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

**Note that** #

If $f(x)$ is continuous & $g(x)$ is discontinuous at $x=a$ then the product function $\phi(x)=f(x)\cdot g(x)$ will be necessarily continuous at $x=a$ provided $f(a)=0$.

$$h(x) = \underbrace{(\sin x)}_{\text{cont}}\underbrace{[x]}_{\text{discont}} \text{ at } x=0$$

$h(x) \to$ cont at $x=0$ since $\sin 0 = 0$

$h(x) = \cos x\,[x]$ at $x=0$

$h(x)$ is discont at $x=0$

$f(x) = (x-1)[x]$

at $x=1$

$f(x)$ will become cont

$$h(x) = \underbrace{x}_{\text{cont}} \underbrace{\sin(1/x)}_{\text{discont}} \quad \text{at } x = 0$$

$h(x)$ will become cont at $x=0$

## QUESTION [JEE Main-2019 (January)]

[Ans. 8]

Let $f(x) = x\left[\frac{x}{2}\right]$ for $-10 < x < 10$, where $[t]$ denotes the greatest integer function. Then the number of points of discontinuity of f is equal to ............... .

$$-5 < \frac{x}{2} < 5$$

$$x/2 \rightarrow \underbrace{1, 2, 3, 4, -1, -2, -3, -4}_{\text{discont}}, 0$$

QUESTION [JEE Main-2019 (January)]

$\sin n\pi = 0$

$\lim_{t \to \infty} \frac{[t]}{t} \to 1$ [Ans. B]

Let [t] denote the greatest integer ≤ t and $\lim_{x \to 0} x\left[\frac{4}{x}\right] = A$. Then the function, $f(x) = [x^2]\sin(\pi x)$ is discontinuous, when x is equal to :

(A) $\sqrt{A} = \sqrt{4} = 2$

(B) $\sqrt{A+1} = \sqrt{5}$

(C) $\sqrt{A+5} = 3$

(D) $\sqrt{A+21} = 5$

$x^2 \to \text{Int}$

$\lim_{x \to 0} x\left[\frac{4}{x}\right] = A$

$\frac{4}{x} = t \Rightarrow x = \frac{4}{t}$

$\lim_{t \to \infty} 4\frac{[t]}{t} \to 1$

$= \boxed{4 = A}$

## QUESTION [IIT JEE-2012]

[Ans. A, B, D]

Let [x] be the greatest integer less than or equals to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous?

(A) $x = -1$

(B) $x = 1$

(C) $x = 0$

(D) $x = 2$

QUESTION

HW

If $f(x) = \frac{\sin 2x + A \sin x + B \cos x}{x^3} (x \neq 0)$ is continuous at $x = 0$. Find the values of A and B. Also find $f(0)$.

## QUESTION [JEE MAIN 2022 (Jul.-II)]

[Ans. B]

HW

If for $p \neq q \neq 0$, then function

$f(x) = \frac{\sqrt[7]{p(729+x)}-3}{\sqrt[3]{729+qx}-9}$ is continuous at $x = 0$, then:

(A) $7pq\, f(0) - 1 = 0$  $\cancel{7} \times 3 \times \frac{1}{\cancel{7}} - 1$

(B) $63q\, f(0) - p^2 = 0$ ✓  $\cancel{63}^{9} \times \frac{1}{\cancel{7}} - 3^2$  $f(0) = 1/7$

(C) $21q\, f(0) - p^2 = 0$

(D) $7pq\, f(0) - 9 = 0$

$$\lim_{x\to 0} f(x) = f(0)$$

$$\lim_{x\to 0}\left(\frac{[p(729+x)]^{1/7} - 3}{(729+qx)^{1/3} - 9}\right) = f(0)$$

Form $\frac{[p(729)]^{1/7} - 3}{(729)^{1/3} - 9} = f(0)$

$[p(729)]^{1/7} - 3 \to 0$, $(729)^{1/3} - 9 \to 0$

$\boxed{729 = 9^3 = 3^6}$

$D^r \to 0 \Rightarrow N^r \to 0$

$(p(729))^{1/7} = 3$

$p(729) = 3^7$

$p \cdot 3^6 = 3^7$

✓ $\boxed{p = 3}$

$$\lim_{x\to 0} \frac{[p(729+x)]^{1/7}-3}{(729+9x)^{1/3}-9}$$

$$\lim_{x\to 0} \frac{\frac{1}{7}(p.(729+x))^{-6/7} \cdot p}{\frac{1}{3}(729+9x)^{-2/3} \cdot 9}$$

$$\frac{3}{7}\frac{(p.729)^{-6/7} \cdot p}{(729)^{-2/3} \cdot 9} = f(0)$$

$$\frac{3}{7}\frac{[3^{\cancel{7}}]^{-6/\cancel{7}} \cdot 3}{(3^{\cancel{6}})^{-2/\cancel{3}} \cdot 9} = f(0)$$

$$\frac{\cancel{3}}{7}\frac{\cancel{3^{-6}}}{\cancel{3^{-4}}\,9}\cancel{3} = f(0)$$

$$\frac{1}{79} = f(0)$$

$$\boxed{p = 3}$$

$$\boxed{9 f(0) = 1/7}$$

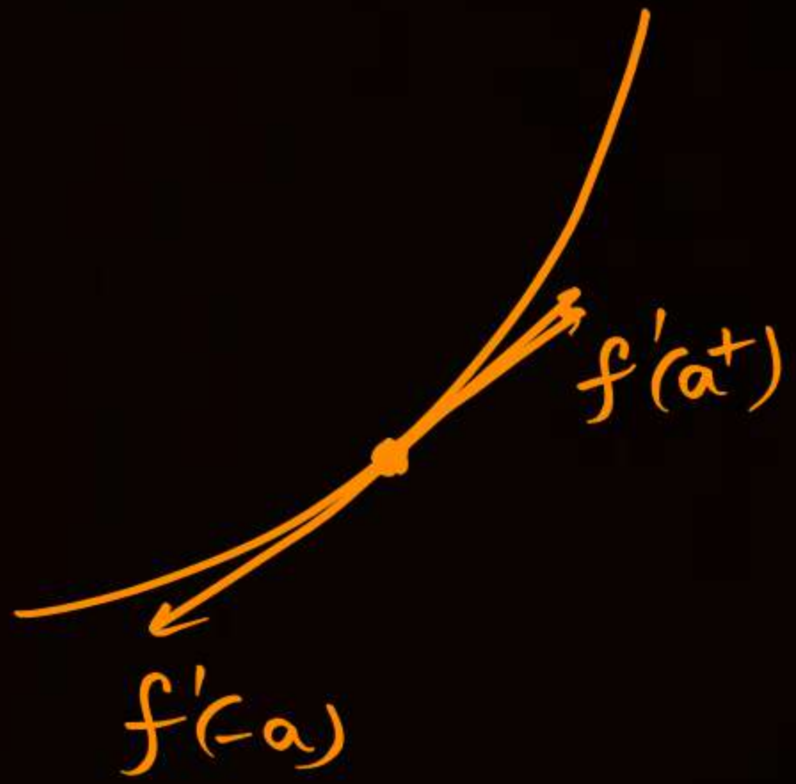
# Differentiability at a point $x = a$

$$\underbrace{LHD} = \underbrace{RHD} = \text{finite}$$

$$f'(a^-) = f'(a^+) = \text{finite}$$

$$RHD = f'(a^+) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$LHD = f'(a^-) = \lim_{h \to 0} \left( \frac{f(a-h) - f(a)}{-h} \right)$$



## Note that

**Right hand & Left hand Derivatives:**

(i) The right hand derivative of $f'$ at $x = a$ denoted by $\boldsymbol{f'(a^+)}$ is defined by :

$$\boldsymbol{f'(a^+) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}},$$

provided the limit exists & is finite.

(ii) The left hand derivative of $f$ at $x = a$ denoted by $\boldsymbol{f'(a^-)}$ is defined by :

$\boldsymbol{f'(a^-) = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}}$, Provided the limit exists and is finite.

$\boldsymbol{f'(a)}$ exists if and only if LHD = RHD = finite

QUESTION

Discuss the differentiability of $f(x)=x|x|$

$\begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$

at $\boxed{x=0}$

RHD: $f'(0^+) = \dfrac{f(0+h)-f(0)}{h} = \dfrac{(0+h)^2-0}{h} = \dfrac{h^2}{h}$

$\lim_{h\to 0} = h$

$=0$

LHD $= f'(0^-) = \dfrac{f(0-h)-f(0)}{-h} = \dfrac{+(0-h)^2-0}{+h}$

$= \dfrac{h^2}{h} = h = 0$

$\boxed{\text{diff at } x=0}$

$\boxed{f'(0)=0}$

## QUESTION

$$f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$ check the differentiability at $x = 0$.

$e^{1/h} \rightarrow$

$e^{-1/h} \rightarrow$

$e^{-\infty} \rightarrow 0$

$$f'(0^+) = \frac{f(0+h) - f(0)}{h}$$

$$= \frac{\dfrac{h}{1+e^{1/h}} - 0}{h}$$

$$= \frac{\cancel{h}}{(1+e^{1/h})\cancel{h}} = \frac{1}{1+e^{1/h}}$$

$$\lim_{h\to 0} \frac{1}{1+e^{1/h}} = \frac{1}{\infty} = \boxed{0}$$

$$f'(0^-) = \frac{f(0-h) - f(0)}{-h}$$

$$= \frac{\dfrac{\cancel{-h}}{1+e^{-h}} - 0}{\cancel{-h}}$$

$$= \frac{1}{1+e^{-1/h}} \rightarrow -\infty$$

$$= \frac{1}{1+0} = 1$$

QUESTION

\# ✓

If the function f(x) defined as $f(x)=\begin{cases}-\dfrac{x^2}{2} & \text{for } x\le 0\\ x^n\sin\dfrac{1}{x} & \text{for } x>0\end{cases}$ is continuous but not derivable at $x = 0$ then find the range of n.

LHL = 0

RHL $\lim_{x\to 0^+} \underbrace{\underbrace{x^n}_{0^n} \underbrace{\sin(1/x)}_{\text{osci}}}_{\downarrow\, 0}$

For continuity $\boxed{n>0}$ ✓

LHD: $f'(0^-) = \dfrac{f(0-h)-f(0)}{-h} = \dfrac{+\frac{(0-h)^2}{2} - 0}{+h} = \dfrac{\frac{h^2}{2}}{h} = h/2 = 0$

RHD: $f'(0^+) = \dfrac{f(0+h)-f(0)}{h} = \dfrac{h^n \sin\frac{1}{h}}{h} = \underbrace{h^{n-1}}_{0^{n-1}} \times \underbrace{\sin(1/h)}_{[-1,1]} \ne 0$

for RHD $\neq 0$

$$\Rightarrow 0^{n-1} \neq 0$$

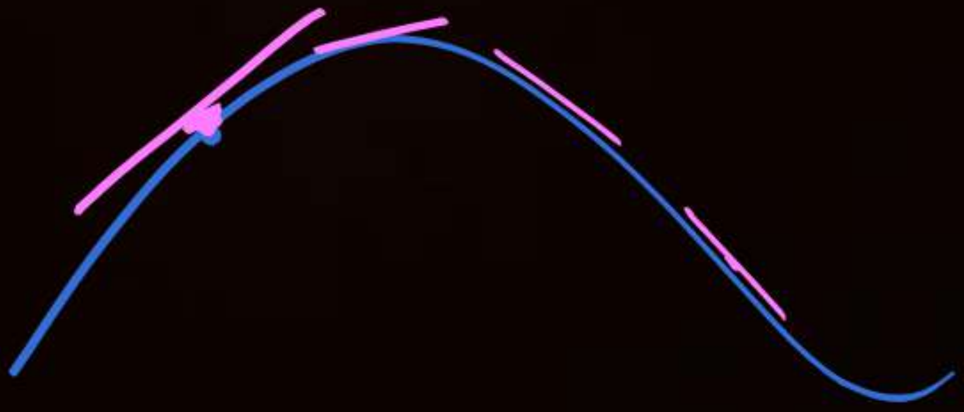
$$\Rightarrow \boxed{n-1 \leq 0}$$

$$\boxed{n \leq 1}$$

Ans $n \in (0, 1]$

# Geometrical Interpretation of Differentiability

If a function $y = f(x)$ is differentiable at $x = a$, then the graph of $y = f(x)$ will have a unique non-vertical tangent at $x = a$.



# Continuity vs Differentiability

(i) If $f'(a)$ exists then $f(x)$ is continuous at $x = a$.

(ii) If $f(x)$ is derivable for every point of its domain of definition, then it is continuous in that domain.

The Converse of the above result is not true i.e.

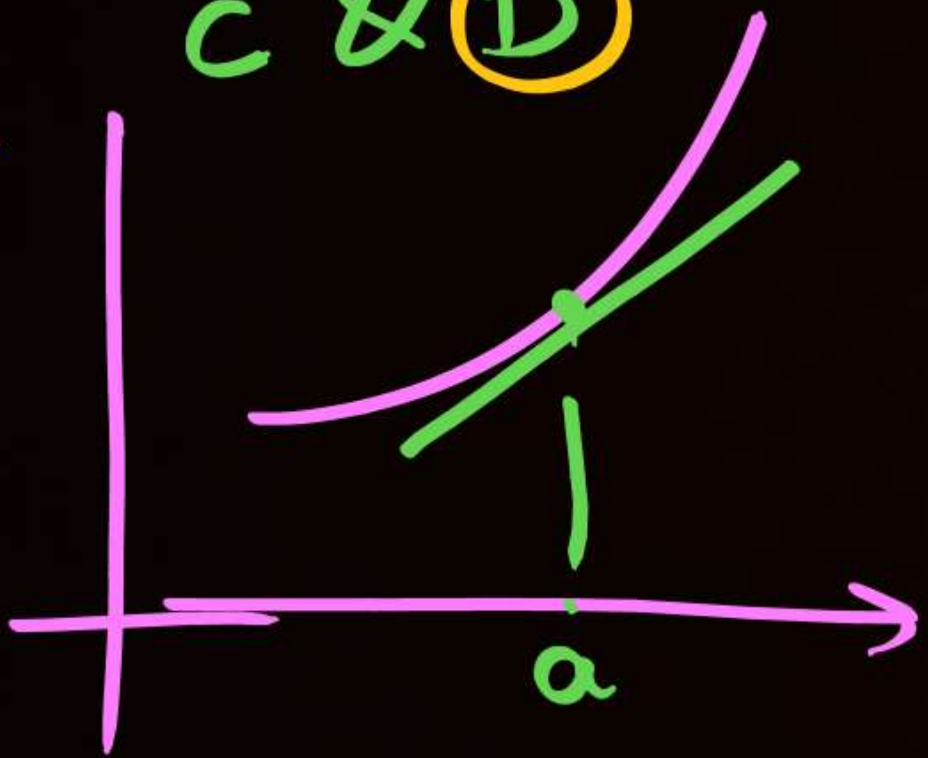
"If $f'$ is continuous at $x = a$, then 'f' is derivable at $x = a$" is not true. e.g. he functions $f(x) = |x - 2|$ is continuous at $x = 2$ but not derivable at $x = 2$.

(iii) If a function f is not differentiable but is continuous at $x = a$ it geometrically implies a sharp corner or kink at $x = a$.

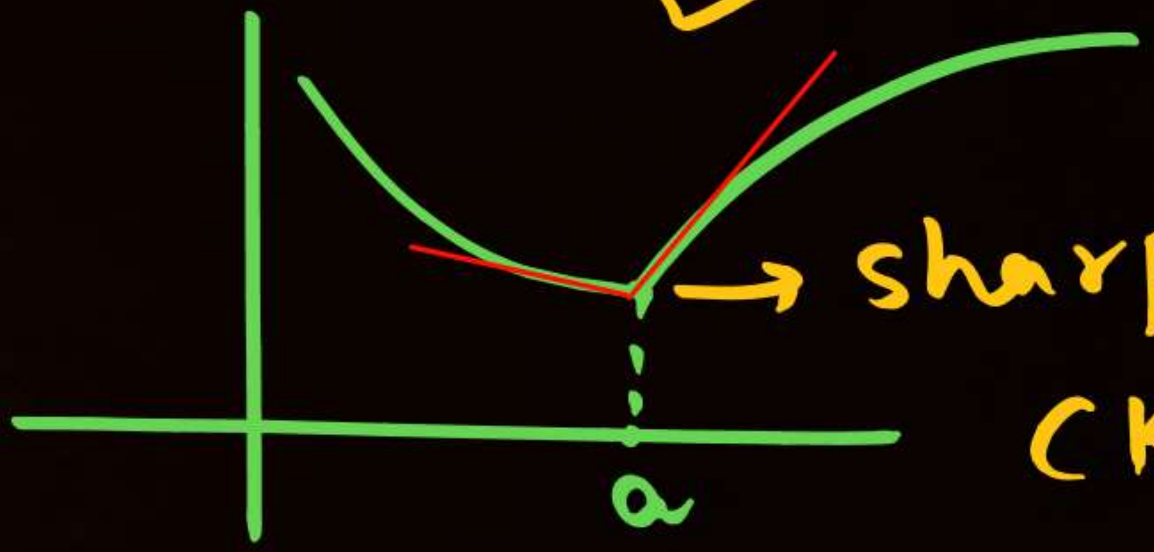
# Continuity Vs Differentiability

(C) (D)

1) C & D

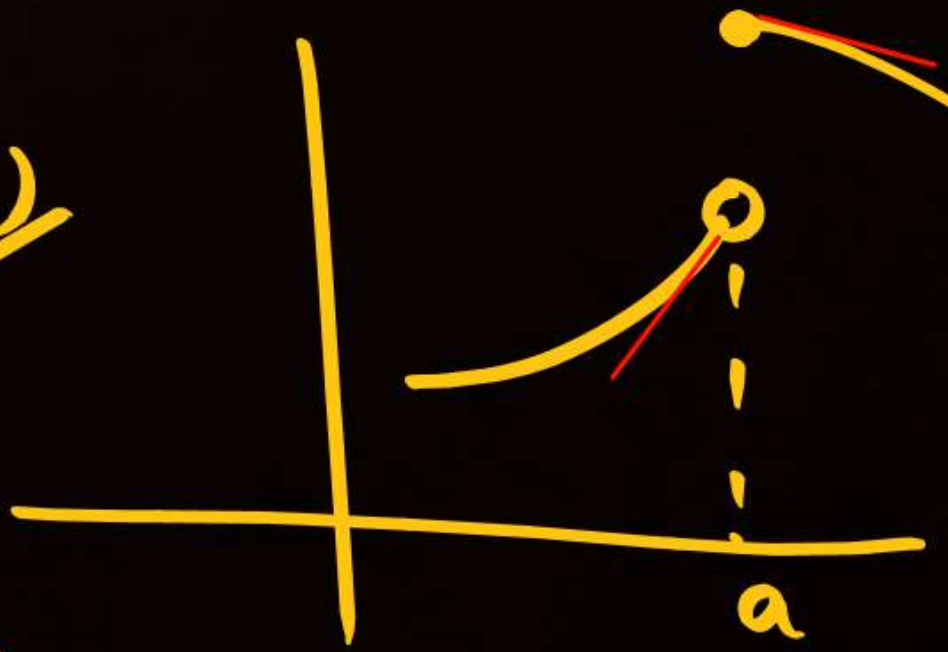



2) C & $\bar{D}$




→ sharp cornor (Kink)

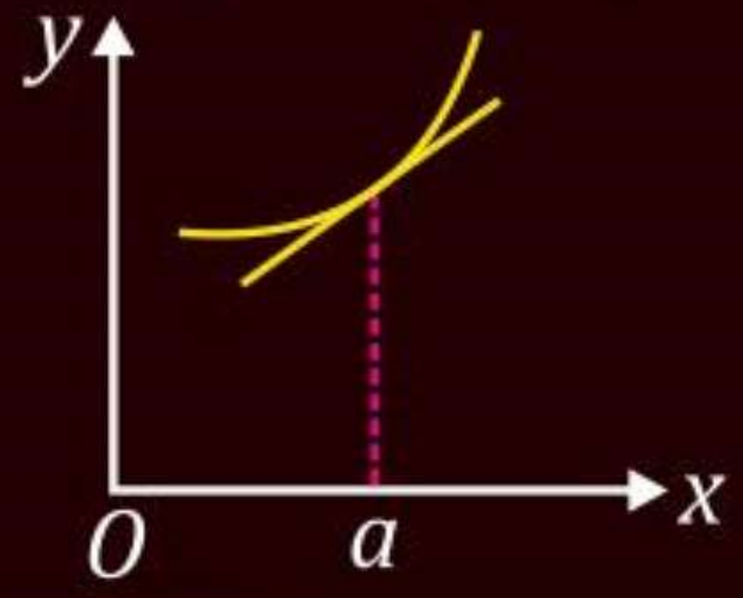
3)



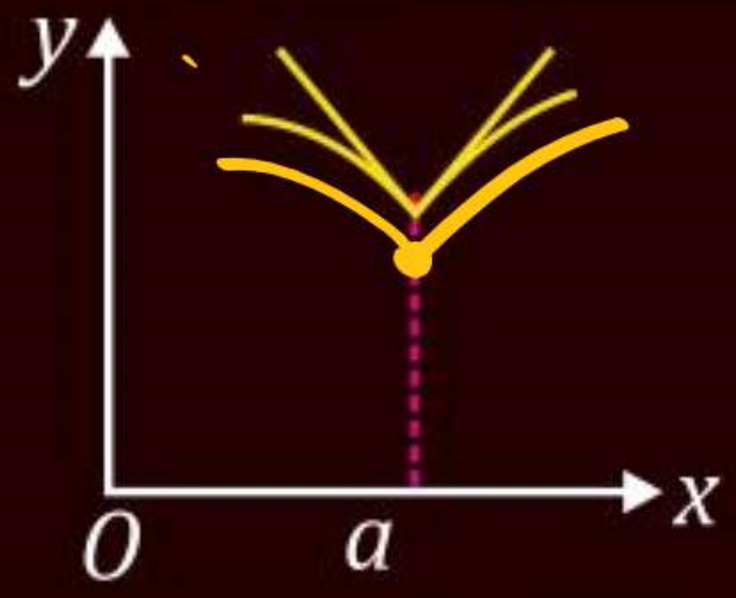

$(\bar{C} \ \& \ \bar{D})$

$\cancel{\bar{C} \ \& \ D}$ → not possible

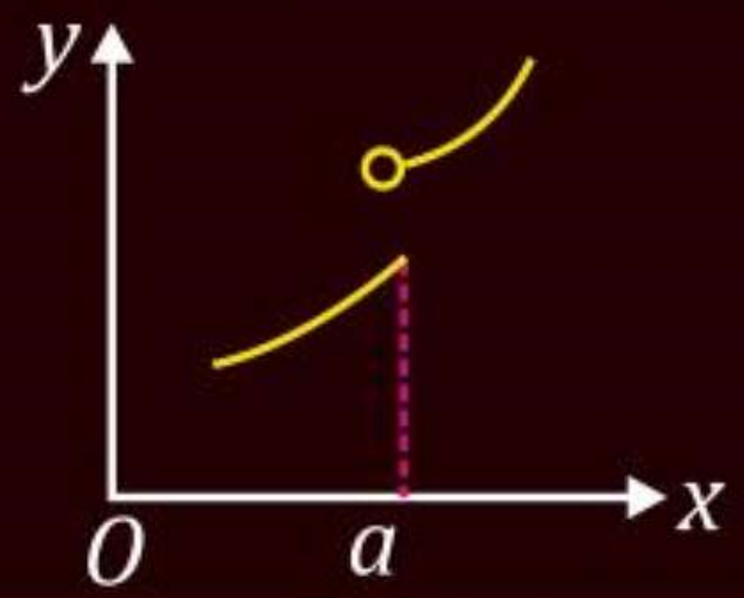
## Note that




(i) Continuous and differentiable




(ii) Continuous but not differentiable




(iii) Neither continuous nor differentiable

# Summary

Differentiability $\Rightarrow$ Continuity

Discontinuity $\Rightarrow$ Non differentiability

Non differentiability $\nRightarrow$ Discontinuity

Continuity $\nRightarrow$ Differentiability

QUESTION [2024 JAN]

If the function $f(x) = \begin{cases} \frac{1}{|x|}, & |x| \geq 2 \\ ax^2 + 2b, & \boxed{|x| < 2} \end{cases}$ is differentiable on $\mathbf{R}$, then $48(a + b)$ is equal to ______

$\Rightarrow f(x)$ is also cont at $x = 2$

$LHL = RHL \Rightarrow \frac{1}{2} = a(2)^2 + 2b$

$\frac{1}{2} = 4a + 2b$

$\boxed{2a + b = \frac{1}{4}} \rightarrow (1)$

$-\frac{1}{8} + b = \frac{1}{4} \Rightarrow b = \frac{1}{8} + \frac{1}{4} = 3/8$

$b = 3/8$

$48b = 18$

$48a + 48b$

$-3 + 18$

$= 15$

$f(x) = 1/x \Rightarrow f'(x) = -\frac{1}{x^2}$

$\boxed{f'(2) = -\frac{1}{4}}$

$f(x) = ax^2 + 2b$

$f'(x) = 2ax$

$f'(2) = 4a.$

$LHD = 4a.$

$RHD = -\frac{1}{4}$

$4a = -\frac{1}{4}$

$a = -\frac{1}{16}.$

$48a = -3$

# Differentiability over an Interval

f(x) is said to be differentiable over an open interval if it is differentiable at each & every point of the interval. $(2, 5)$

$[\underbrace{2, 5}]$

**f(x) is said to be differentiable over a closed interval [a, b] if:**

at $x = 2$ check RHD only

(i) for the points a and b, $f'(a^+)$ & $f'(b^-)$ exist finitely.

(ii) It is differentiable at every point of (a, b)

at 5 check LHD only

## QUESTION [JEE Main-2023 (Apr.-I)]

[Ans. B]

$\boxed{m=1}$

$n=0$

Let [x] denote the greatest integer function and $f(x) = \max\{1 + x + [x], 2 + x, x + 2[x]\}$, $0 \le x \le 2$.

Let m be the number of points in [0, 2], where f is not continuous and n be the number of points in (0, 2), where f is not differentiable.

Then $(m + n)^2 + 2$ is equal to

(A) 2

(B) 3

(C) 6

(D) 11

(1) If $0 \le x < 1$

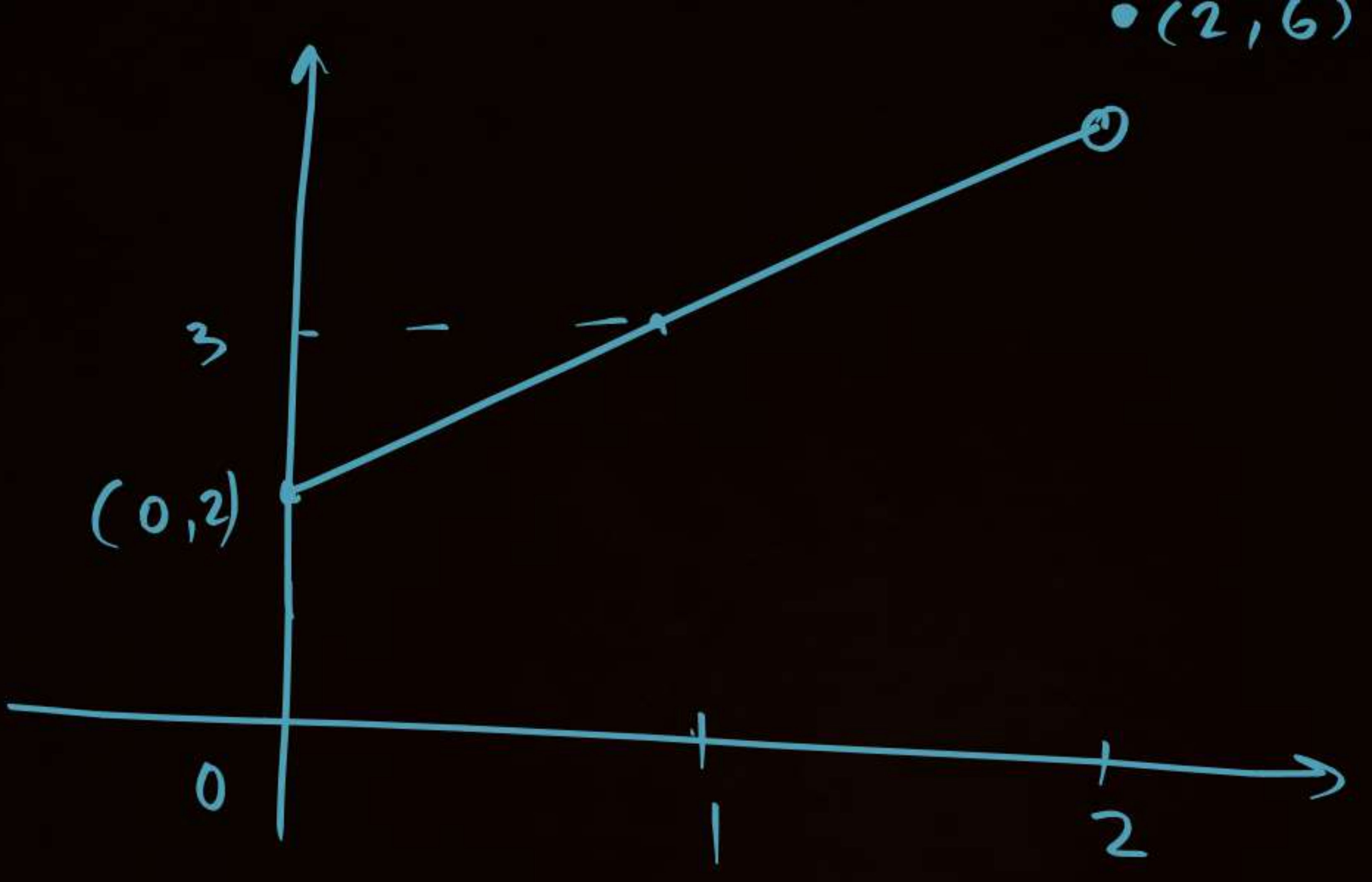
$f(x) = \max\{1+x, 2+x, x\} \Rightarrow \boxed{y = x+2}$

(2) If $1 \le x < 2$

$f(x) = \max\{2+x, 2+x, x+2\} \Rightarrow y = x+2$

(3) If $\boxed{x=2}$

$f(x) = \max\{5, 4, 6\} \Rightarrow y = 6$




discont at $x = 2$

diff every where in $(0, 2)$

QUESTION [JEE Main-2022 (June)]

Let $f(x) = \begin{cases} \dfrac{\sin(x-[x])}{x-[x]}, & x \in (-2,-1) \\ \max\{2x, 3[|x|]\}, & |x| < 1 \\ 1, & \text{otherwise} \end{cases}$

$[x] \to -2$, $[|x|] \to 0$

$f(x) = \begin{cases} \dfrac{\sin(x+2)}{(x+2)}, & x \in (-2,-1) \\ \max\{2x, 0\} & x \in (-1,1) \\ 1 & \text{otherwise} \end{cases}$

where [t] denotes greatest integer ≤t. If m is the number of points where f is not continuous and n is the number of points where f is not differentiable, then the ordered pair (m, n) is

(A) (3, 3)

(B) (2, 4)

(C) (2, 3)

(D) (3, 4)

[Ans.

$f(x) = \begin{cases} \dfrac{\sin(x+2)}{x+2}, & x \in (-2,-1) \\ 2x & \text{if } x \in [0,1) \\ 0 & \text{if } x \in (-1,0) \\ 1 & \text{otherwise} \end{cases}$

at $x=-1$
LHL $= \dfrac{\sin 1}{1}$
RHL $= 0$
discont

at $x=0$ cont

at $x=-2$
RHL $= 1$, LHL $= 1$

at $x=1$
LHL $= 2$ $\Rightarrow$ discont
RHL $= 1$

discont $\Rightarrow$ non diff

$\Rightarrow$ non diff at $x = \pm 1$ ✓

non diff at $x = 0$

# Properties of Differentiable Functions

(i) If $f(x)$ & $g(x)$ are derivable at $x = a$ then the functions $f(x) \pm g(x)$, $f(x).\,g(x)$, $f(x)/g(x)$ $(g(a) \neq 0)$ will also be derivable at $x = a$.

(ii) If $f(x)$ is not differentiable at $x = a$ & $g(x)$ is differentiable at $x = a$, then the functions $f(x) \pm g(x)$ will not be differentiable at $x = a$

(iii) If $f(x)$ is not differentiable at $x = a$ & $g(x)$ is differentiable at $x = a$, then the product function $F(x) = f(x)g(x)$ can still be differentiable at $x = a$

e.g. $f(x) = |x|$ and $g(x) = x$

# Properties of Differentiable Functions

(iv) If $f(x)$ & $g(x)$ both are not differentiable at $x = a$ then the product function; $F(x) = f(x) \cdot g(x)$ can still be differentiable at $x = a$
e.g. $f(x) = |x|$ & $g(x) = |x|$

(v) If $f(x)$ & $g(x)$ both are non-derivable at $x = a$ then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function.
e.g. $f(x) = |x|$ & $g(x) = -|x|$.

# Properties -Summary

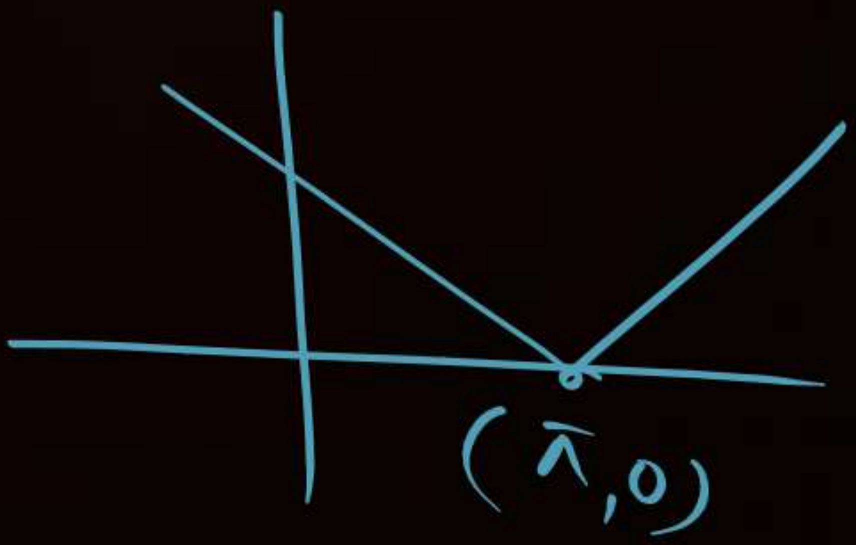
| $f$ | $g$ | $f \pm g$ | $fg$ or $f/g$ |
|---|---|---|---|
| ✓ | ✓ | ✓ | ✓ |
| × | ✓ | × | ? |
| × | × | ? | ? |

## Note that

If $\boldsymbol{f(x)}$ is differentiable & $\boldsymbol{g(x)}$ is non differentiable at $\boldsymbol{x = a}$ then the product function $\boldsymbol{\phi(x) = f(x) \cdot g(x)}$ will be necessarily differentiable at $\boldsymbol{x = a}$ provided $f(a) = 0$.

Ex→1 $f(x) = |x-\pi| \sin x$

is diff at $x = \pi$




Ex→2

$f(x) = |x-\pi| \cos x$

at $x = \pi$ is

non diff

QUESTION

The function $f(x) = (x^2 - 4)|\underbrace{x^2 - 3x + 2}| + \cos|x|$ is not differentiable at $x = ?$

$\cos x$

(A) $-1$

(B) $0$

(C) $1$

(D) $2$

$(x^2-4)\,|\underbrace{(x-2)(x-1)}| + \cos x$

$x = 2$ or $1$

cont $\qquad$ discont

at $x = 1$ $\qquad \underbrace{\text{discont} + \text{cont}}_{\text{discont}}$

QUESTION

Number of points where the function $f(x) = (x^2 - 1)|x^2 - x - 2| + \sin(|x|)$ is not differentiable, is

(A) 0

(B) 1

(C) 2

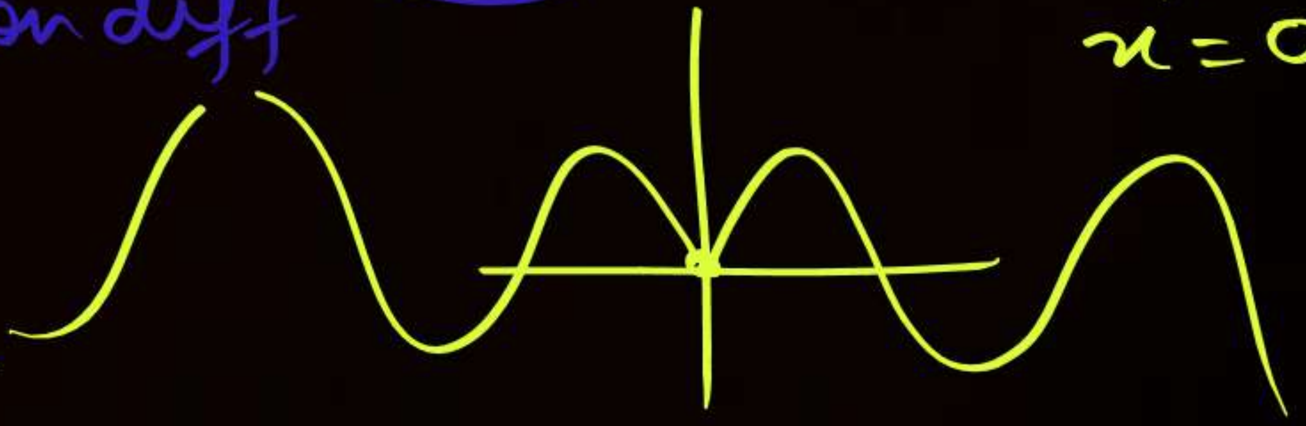
(D) 3

$$(x^2-1)\,\Big|\underbrace{(x-2)(x+1)}\Big| + \underbrace{\sin(|x|)}$$

$x=2$ or $\cancel{x=-1}$

non diff

not diff at $x=0$



$x=0 \rightarrow$ non diff

$x=2 \rightarrow$ non diff

## QUESTION [JEE Main-2022 (July-I)]

[Ans. B]

The number of points, where the function $f : R \to R$,
$f(x) = |x-1|\cos|x-2|\sin|x-1| + (x-3)|x^2-5x+4|$,
is not differentiable, is:

(A) 1

(B) 2

(C) 3

(D) 4

$|x-1|\cos(x-2)\sin|x-1| + \underbrace{(x-3)}\,|(x-4)(x-1)|$

$x=4$ or $x=1$

## QUESTION [JEE Main-2023 (Feb.-I)]

[Ans. 2]

The number of points, at which the function $f(x) = |2x+1| - 3|x+2| + |x^2 + x - 2|, x \in R$ is not differentiable, is ________

$$|(x+2)(x-1)|$$

$$|2x+1| - 3|x+2| + |x+2||x-1|$$

$$|2x+1| + |x+2|\,(|x-1| - 3)$$

$$x = -\tfrac{1}{2} \qquad x = -2 \qquad \boxed{x = 1}$$

$$\overline{D} \qquad\qquad X \qquad\qquad \overline{D}$$

QUESTION [JEE Main-2022]

HW

[Ans. 79]

If [t] denotes the greatest integer $\leq$ t, then number of points, at which the function $f(x) = 4|2x + 3| + 9\left[x + \frac{1}{2}\right] - 12[x + 20]$ is not differentiable in the open interval $(-20, 20)$, is ________.

# Functional Identities

(i) $\boxed{f(xy) = f(x) + f(y)} \Rightarrow \boxed{f(x) = k \ln x}$ or $f(x) = 0$

(ii) $\boxed{f(xy) = f(x) \cdot f(y)} \Rightarrow \boxed{f(x) = x^n}, n \in R$

(iii) $\boxed{f(x + y) = f(x) \cdot f(y)} \Rightarrow \boxed{f(x) = a^x}.$

(iv) $\boxed{f(x + y) = f(x) + f(y)} \Rightarrow \boxed{f(x) = kx}$, where k is a constant.

**QUESTION**

$y=0 \Rightarrow f(\cancel{x}) = \cancel{f(x)}\, f(0)$

$f(0)=1$

If f(x) is a differentiable function satisfying $f(x+y) = f(x)f(y)$ $\forall$ $x, y \in R$ and $f'(0) = 3$. Then find f(x).

$f(x) = a^x$

$f'(x) = a^x \ln a$

$f'(0) = a^0 \ln a$

$3 = \ln a$

$\boxed{a = e^3}$

$\boxed{f(x) = e^{3x}}$

Method →

Step-1 diff wrt x keeping y const.

$f(x+y) = f(x) f(y)$

$f'(x+y) = f(y) \cdot f'(x)$

Put $x = 0$

$f'(y) = f(y) \cdot f'(0)$

$f'(y) = f(y) \cdot 3$

$y \rightarrow x$

$f'(x) = 3f(x) \rightarrow$ D.E.

Step → 2 Integrate D.E.

$f'(x) \rightarrow \frac{dy}{dx}$ $\qquad y \rightarrow f(x)$

$$\frac{dy}{dx} = 3y$$

$$\int \frac{dy}{y} = \int 3\, dx$$

$$\ln y = 3x + c$$

Step → 3 Find 'c'

$$f(0) = 1$$

$$\ln(f(x)) = 3x + c$$

$$\ln 1 = 3 \times 0 + c$$

$$\boxed{0 = c}$$

$$\ln y = 3x$$

$$y = e^{3x}$$

$$\boxed{f(x) = e^{3x}}$$

QUESTION

$\lim_{x \to 0} \frac{f(x)}{x} = 1$ $\quad \frac{f'(x)}{1} = \boxed{f'(0) = 1}$

$\boxed{f(0) = 0.}$

Suppose f is a derivable function that satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y.

If $\boxed{\lim_{x\to 0} \frac{f(x)}{x} = 1}$, find

✓ diff w.r.t $x$ keeping $y$ const

$f'(x+y) = f'(x) + 0 + \cancel{2xy} + y^2 \cdot 1$

$\boxed{x=0}$

$f'(y) = f'(0) + y^2$

$f'(y) = 1 + y^2$

$y \to x$

$f'(x) = 1 + x^2$

$dy/dx = 1 + x^2$

$\int dy = \int (1+x^2)\,dx$

$y = \int dx + \int x^2 dx$

$f(x) = x + x^3/3 + c$

$f(0) = 0$

$\Rightarrow c = 0$ $\quad \boxed{f(x) = \frac{x^3}{3} + x}$

(A) $f(0) \to 0$

(B) $f'(0) = 1$

(C) $f'(x) \to 1 + x^2$

(D) $f(3) \to$

QUESTION [JEE Main-2023 (Jan.)]

put $y=0$ $5f(\cancel{x}) = \cancel{f(x)} f(0)$ $\quad f(x) \to y$

Suppose $f : R \to (0, \infty)$ be a differentiable function such that $5f(x + y) = f(x) \cdot f(y)$, $\forall x, y \in R$. If $f(3) = 320$, then $\sum_{n=0}^{5} f(n)$ is equal to:

(A) 6875

(B) 6575

(C) 6825

(D) 6528

$5f(x+y) = f(x)f(y)$

diff wrt $x$

$5f'(x+y) = f'(x)f(y)$

put $x=0$

$5f'(y) = \underbrace{f'(0)}_{K} f(y)$

$5f'(y) = Kf(y)$

$y \to x$

$5f'(x) = Kf(x)$

$5\frac{dy}{dx} = Ky$

$\int \frac{dy}{y} = \int \frac{K\,dx}{5}$

$\ln y = \frac{K}{5}x + c$

$x=0, y=5$

$\ln 5 = 0 + c \Rightarrow c = \ln 5$

$\ln y = \frac{Kx}{5} + \ln 5$

$$\ln(y/5) = \frac{kx}{5}$$

$$y/5 = e^{kx/5}$$

$$y = 5e^{kx/5}$$

$$x = 3,\ y = 320$$

$$320 = 5e^{3k/5}$$

$$64 = e^{3k/5}$$

$$\boxed{4 = e^{k/5}}$$

$$y = 5e^{kx/5}$$

$$5\left(e^{k/5}\right)^x$$

$$\boxed{y = 5(4)^x}$$

$$\sum_{n=0}^{5} 5(4)^n$$

$$5\left[4^0 + 4^1 + \cdots + 4^5\right]$$

## QUESTION [JEE Main-2022 (June-I)]

Let f: N → R be a function such that $\boxed{f(x+y) = 2f(x)f(y)}$ for natural numbers x and y. If $\underbrace{f(1) = 2}$, then the value of $\alpha$ for which

HW

$$\sum_{k=1}^{10} f(\alpha + k) = \frac{512}{3}(2^{20} - 1)$$

holds, is

## QUESTION [JEE Main-2023 (Jan.-I)]

[Ans. 3]

$y=0$ $\quad f(\cancel{x}) = f(\cancel{x}) + f(0) - 1$

$f(0) = 1$

Let f: $\mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the relation

$\boxed{f(x+y) = f(x) + f(y) - 1}$, $\forall x, y \in \mathbb{R}$.

If $\underbrace{f'(0) = 2}$, then $|f(-2)|$ is equal to 3 ____

$f'(x+y) = f'(x)$

$x = 0$

$f'(y) = f'(0)$

$f'(y) = 2$

$y \to x$

$f'(x) = 2$

$dy/dx = 2$

$\int dy = \int 2\,dx$

$y = 2x + c$

$x = 0,\ y = 1$

$1 = 2 \times 0 + c$

$\boxed{c = 1}$

$\boxed{y = 2x + 1}$

$f(x) = 2x + 1$

$f(-2) = -4 + 1 = -3$

## QUESTION [JEE Main-2023)]

Let $f: R \to R$ satisfy the equation $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in R$ and $f(x) \neq 0$ for any $x \in R$. If the function f is differentiable at $x = 0$ and $f'(0) = 3$, then $\lim_{h \to 0} \frac{1}{h}(f(h) - 1)$ is equal to ______

HW

# Some Common Derivatives

| | f (x) | f ′(x) |
|---|---|---|
| (i) | $x^n$ | $nx^{n-1}$ |
| (ii) | $e^x$ | $e^x$ |
| (iii) | $a^x$ | $a^x \ln a, a > 0$ |
| (iv) | $\ln x$ | $1/x$ |
| (v) | $\log_a x$ | $(1/x)\log_a e, a > 0, a \neq 1$ |
| (vi) | $\sin x$ | $\cos x$ |
| (vii) | $\cos x$ | $-\sin x$ |
| (viii) | $\tan x$ | $\sec^2 x$ |
| (ix) | $\sec x$ | $\sec x \tan x$ |
| (x) | $\operatorname{cosec} x$ | $-\operatorname{cosec} x \cdot \cot x$ |
| (xi) | $\cot x$ | $-\operatorname{cosec}^2 x$ |
| (xii) | constant | 0 |

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

# Methods of Differentiation

1. Logarithmic Differentiation ✓
2. Parametric Differentiation ✓
3. Differentiation of Implicit Function ✓
4. Differentiation Using Substitution ✓

$\frac{d}{dx} x^5 \rightarrow 5x^4$

$\frac{d}{dx} (5)^x \rightarrow 5^x \ln 5$

$\frac{d}{dx} (x)^x$ $\quad \boxed{x^y = e^{y \ln x}}$

$y = x^x$

$\ln y = x \ln x$

## QUESTION [JEE Main-2023]

If $y(x) = x^x, x > 0$, then $y''(2) - 2y'(2)$ is equal to

(A) $4(\log_e 2)^2 - 2$

(B) $8\log_e 2 - 2$

(C) $4(\log_e 2)^2 + 2$

(D) $4\log_e 2 + 2$

$4((\ln 2)^2 - 1) + 2$

$4(\ln 2)^2 - 2$

$\ln y = x \ln x$

diff

$\frac{1}{y}\frac{dy}{dx} = \cancel{x} \times \frac{1}{\cancel{x}} + \ln x$

$\frac{y'}{y} = 1 + \ln x$

$y' = y + y\ln x$

$y'(2) = 4 + 4\ln 2 = 4 + 4\ln 2$

diff $y'' = y' + y \times \frac{1}{x} + (\ln x)y'$

$y'' = y' + y/x + (\ln x)y'$

$\boxed{x = 2}$, $y = 2^2$

$y'' = (4 + 4\ln 2) + 2 + (\ln 2)(4 + 4\ln 2)$

$y'' = (4 + 4\ln 2)(1 + \ln 2) + 2$

$2y'(2) = 2(4 + 4\ln 2)$

$y'' - 2y' =$

$(4 + 4\ln 2)(\ln 2 - 1) + 2$

$4(\ln 2 + 1)(\ln 2 - 1) + 2$

QUESTION [JEE Main-2023]

If $2x^y + 3y^x = 20$, then $\boxed{\frac{dy}{dx} \text{ at } (2, 2)}$ is equal to

(A) $-\left(\frac{3+\log_e 16}{4+\log_e 8}\right)$

(B) $-\left(\frac{3+\log_e 4}{2+\log_e 8}\right)$

(C) $-\left(\frac{2+\log_e 8}{3+\log_e 4}\right)$ ✓

(D) $-\left(\frac{3+\log_e 8}{2+\log_e 4}\right)$

$x^y = e^{y\ln x}$

$2e^{y\ln x} + 3e^{x\ln y} = 20$

diff wrt $x$

$2e^{y\ln x}\left[\frac{y}{x} + y'\ln x\right] + 3e^{x\ln y}\left[\ln y + x\cdot\frac{1}{y}y'\right] = 0$

$2(x)^y\left[\frac{y}{x} + y'\ln x\right] + 3x^y\left[\ln y + \frac{xy'}{y}\right] = 0$

$x=2,\ y=2$

$2\times\cancel{4}\left[1 + y'\ln 2\right] + 3\times\cancel{4}\left[\ln 2 + y'\right] = 0$

$2 + 2y'\ln 2 + 3\ln 2 + 3y' = 0$

$y'(2\ln 2 + 3) = -2 - 3\ln 2$

$y' = -\frac{(2+3\ln 2)}{2\ln 2 + 3}$

# Parametric Differentiation

Suppose y and x are two functions of θ such that $\underbrace{y = f(\theta)}$ and $\underbrace{x = g(\theta)}$ where "θ" is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$.

$$y = 2\sin 2\theta \quad , \quad x = 3\cos 3\theta$$

$$dy/d\theta = 2\cos 2\theta \cdot 2 \qquad \frac{dx}{d\theta} = -3^{2}\sin 3\theta$$

$$dy/dx = \frac{4\cos 2\theta}{-9\sin 2\theta}$$

QUESTION [JEE Main 2023]

[Ans.

If $f(x) = x^2 + g'(1)x + g''(2)$ and $g(x) = f(1)x^2 + xf'(x) + f''(x)$, the value of $f(4) - g(4)$ is equal to ________.

HW

QUESTION [JEE Main 2023 (Jan. II)]

If $f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3), x \in R$, then

HW

A) $3f(1) + f(2) = f(3)$

B) $f(3) - f(2) = f(1)$

C) $2f(0) - f(1) + f(3) = f(2)$

D) $f(1) + f(2) + f(3) = f(0)$

# Differentiation of 1st function w.r.t. 2nd Function

$\sin x$

diff $\sin x$ wrt $e^x$

Ans $= \frac{\cos x}{e^x}$

$\tan x$ wrt $x^3$

Ans $\frac{\sec^2 x}{3x^2}$

$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x)$

$z = g(x) \Rightarrow dz/dx = g'(x)$

$$\boxed{\frac{dy}{dz} = \frac{f'(x)}{g'(x)}}$$

## QUESTION [JEE Main 2020]

The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x=\frac{1}{2}$ is:

[Ans. C]

(A) $\frac{2\sqrt{3}}{3}$

(B) $\frac{2\sqrt{3}}{5}$

(C) $\frac{\sqrt{3}}{10}$ ✓

(D) $\frac{\sqrt{3}}{12}$

$x = \tan\theta \Rightarrow \tan^{-1}x = \theta$

$y = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$\frac{\cancel{2}\sin^2\theta/2}{\cancel{2\sin\theta/2}\cos\theta/2}$

$\tan^{-1}(\tan\theta/2) = \theta/2$

$\boxed{y = \frac{\tan^{-1}x}{2}}$

$\frac{dy}{dx} = \left(\frac{1}{1+x^2}\right)\cdot\frac{1}{2}$

$= \left(\frac{1}{1+\frac{1}{4}}\right)\cdot\frac{1}{2} = \frac{4}{5}\cdot\frac{1}{2}$

$= \boxed{2/5}$

$z = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ $\quad x = \sin\phi$

$z = \tan^{-1}\left(\frac{2\sin\phi\cos\phi}{\cos 2\phi}\right) = \tan^{-1}\tan 2\phi$

$= 2\sin^{-1}x$

$$z = 2\sin^{-1}x$$

$$\frac{dz}{dx} = \frac{2}{\sqrt{1-x^2}} = \frac{2}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\sqrt{3}} \times 2 = \frac{4}{\sqrt{3}}$$

$$\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{2/5}{4/\sqrt{3}} = \frac{2}{5 \times 4}\sqrt{3}$$

$$= \frac{\sqrt{3}}{10}$$

## QUESTION [JEE Main]

$\frac{d^2x}{dy^2}$ equals

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

(B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

(D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ ✓

[Ans.

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$$

diff wrt y

$$\underbrace{\frac{d}{dy}\left(\frac{dx}{dy}\right)} = \frac{d}{dy}\left(\frac{dy}{dx}\right)^{-1}$$

$$\frac{d^2x}{dy^2} = -1\left(\frac{dy}{dx}\right)^{-2}\frac{d}{dy}\left(\frac{dy}{dx}\right)\frac{dx}{dx}$$

$$= -\left(\frac{dy}{dx}\right)^{-2}\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dy} \qquad \frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

$$= -\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-1}$$

#

$$\boxed{\frac{d^2x}{dy^2} = -\left(\frac{dy}{dx}\right)^{-3}\frac{d^2y}{dx^2}}$$

**QUESTION [Main June 27, 2022 (II)]** $-4$ $16$ [Ans. 16]

If $y(x) = (x^x)^x,\ x > 0$, then $\frac{d^2x}{dy^2} + 20$ at $x = 1$ is equal to

$y = x^{x^2}$ at $x=1$, $y=1$, $y'=1$

$\ln y = x^2 \ln x$

$\frac{1}{y} y' = x^2 \cdot \frac{1}{x} + 2x \ln x$

$\frac{y'}{y} = x + 2x\ln x$

$y' = yx(1 + 2\ln x)$

$y' = xy + 2x\cancel{\ln x} \cdot y$

$x = 1 \Rightarrow \boxed{y' = 1}$

$y'' = xy' + y + \cancel{2x\ln x}\, y' + y\left[2\cancel{x} \cdot \frac{1}{\cancel{x}} + 2\cancel{\ln x}\right]$

$y'' = 1 + 1 + 1 \times 2$

$\boxed{y'' = 4}$

$\frac{d^2x}{dy^2} = -\left(\frac{dy}{dx}\right)^{-3} \frac{d^2y}{dx^2}$

$= -(y')^{-3}(y'')$

$= -(1)^{-3}(4) = -4$

## Differentiation mixed with Inverse

$f' \longrightarrow$

$\boxed{f\, f^{-1}(x) = x}$

#Q. If g is inverse of f and $f'(x) = \frac{1}{1+x^{2024}}$, then show that $g'(x)$ equals $\underbrace{1 + [g(x)]^{2024}}$.

$x \to g(x)$

$f'(g(x)) = \frac{1}{1+(g(x))^{2024}}$

$f(g(x)) = x$

diff wrt $x$

$f'(g(x)) \cdot g'(x) = 1$

$g'(x) = \frac{1}{f'(g(x))}$

$\boxed{g'(x) = 1 + (g(x))^{2024}}$

QUESTION

$f'(x) = e^x + 1$

$f'(\ln 2) = e^{\ln 2} + 1 = 2 + 1 = 3$

The function $\boxed{f(x) = e^x + x,}$ being differentiable and one to one, has a differentiable inverse $f^{-1}(x)$. The value of $\frac{d}{dx}(f^{-1})$ at the point $f(\ell n2)$ is

(A) $\frac{1}{\ell n\, 2}$

(B) $\frac{1}{3}$ ✓

(C) $\frac{1}{4}$

(D) none

$f^{-1} \to g$

Consider

$g'(f(\ln 2)) = ?$

$g(f(x)) = x$

$g'(f(x)) \cdot f'(x) = 1$

$g'f(x) = \frac{1}{f'(x)}$

$x \to \ln 2$

$g'f(\ln 2) = \frac{1}{f'(\ln 2)} = 1/3$

QUESTION

HW

[Ans. A]

If the function $f(x) = -4e^{\frac{1-x}{2}} + 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$ and $g(x) = f^{-1}(x)$, then the value of $g'\left(-\frac{7}{6}\right)$ equals:

(A) $\frac{1}{5}$

(B) $-\frac{1}{5}$

(C) $\frac{6}{7}$

(D) $-\frac{6}{7}$

QUESTION [JEE Main 2024 (Jan. II)]

$2x+3 = 5$
$b = b$
$\boxed{b=5}$

Let $a$ and $b$ be real constants such that the function f defined by

$$f(x) = \begin{cases} x^2+3x+a, & x \le 1 \\ bx+2, & x > 1 \end{cases}$$ be differentiable on $\mathbb{R}$.

Then, the value of $\int_{-2}^{2} f(x)\,dx$ equals

(A) 21

(B) 19/6

(C) 15/6

(D) 17

LHL = RHL

$1+3+a = b+2$

$\boxed{a+2=b} \rightarrow (1)$

$a+2=5$

$\boxed{a=3}$

$$\int_{-2}^{1}(x^2+3x+3)\,dx + \int_{1}^{2}(5x+2)\,dx$$

$$\frac{x^3}{3}+\frac{3x^2}{2}+3x\Big|_{-2}^{1} + \frac{5x^2}{2}\Big|_1^2 + 2x\Big|_1^2$$

$$\frac{1-(-8)}{3}+\frac{3}{2}(1-4)+3(3)+\frac{5}{2}(4-1)+2$$

$$3-\frac{9}{2}+9+\frac{15}{2}+2$$

$$14+\frac{6}{2} = 17$$

## QUESTION [JEE Main 2024 (Jan. II)]

Consider the function $f:(0,\infty)\to\mathbb{R}$ defined by $f(x)=e^{-|\log_e x|}$. If m and n be respectively the number of points at which f is not continuous and f is not differentiable, then m + n is

$\boxed{m=0}$

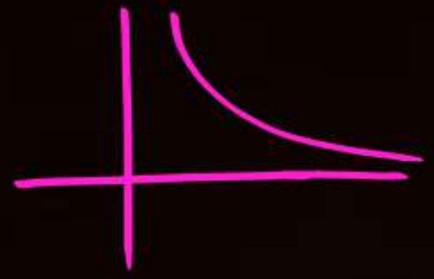
$\boxed{n=1}$

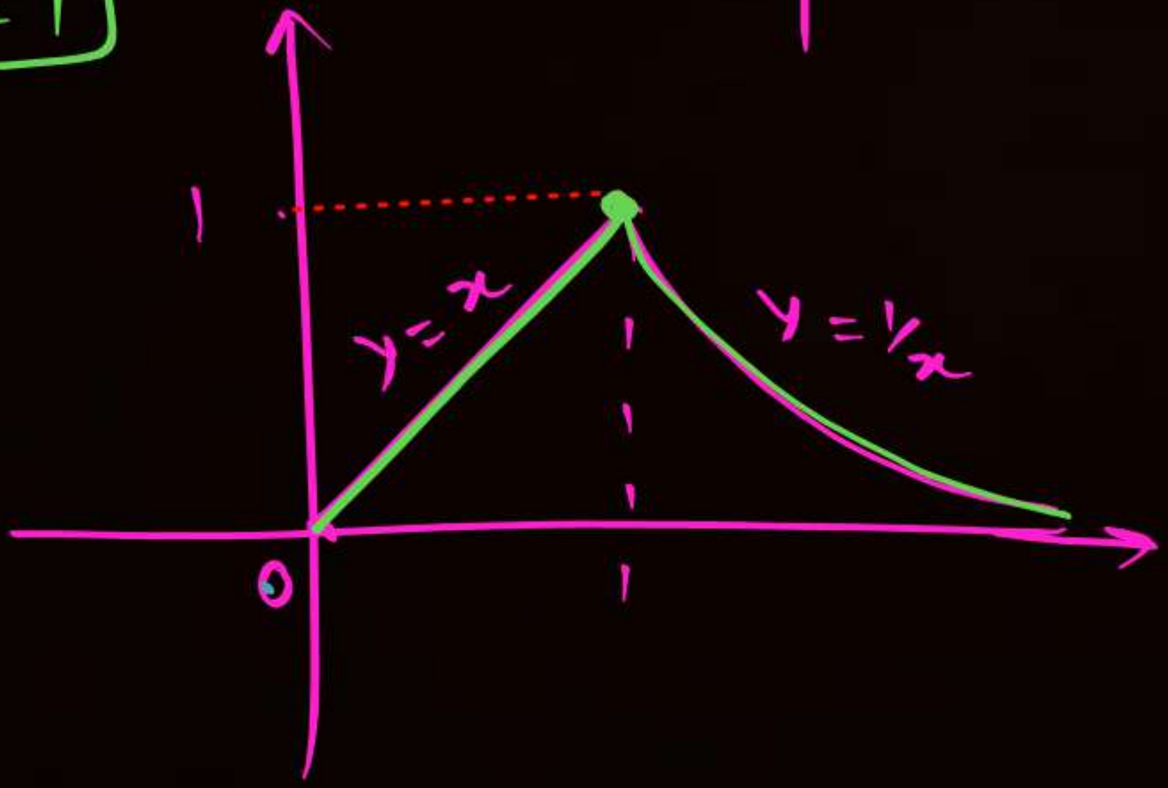
(A) 1 ✓

(B) 3

(C) 2

(D) 0






(1) $x>1$

$\Rightarrow \ln x>0$

$y=e^{-\ln x}=\frac{1}{x}$

(2) $0<x<1$

$\ln x<0$

$y=e^{\ln x}$

$\boxed{y=x}$

## QUESTION [JEE Main 2024 (Feb. II)]

$m = 0$
$n = 3$

Let $\boxed{f(x) = |2x^2 + 5|x| - 3|}$, $x \in \mathbf{R}$. If m and n denote the number of points where $f$ is not continuous and not differentiable respectively, then $m + n$ is equal to :

$x^2 = |x|^2$

(A) 0

$f(x) = \left|2|x|^2 + 5|x| - 3\right|$

(B) 2

$g(x) = |2x^2 + 5x - 3|$

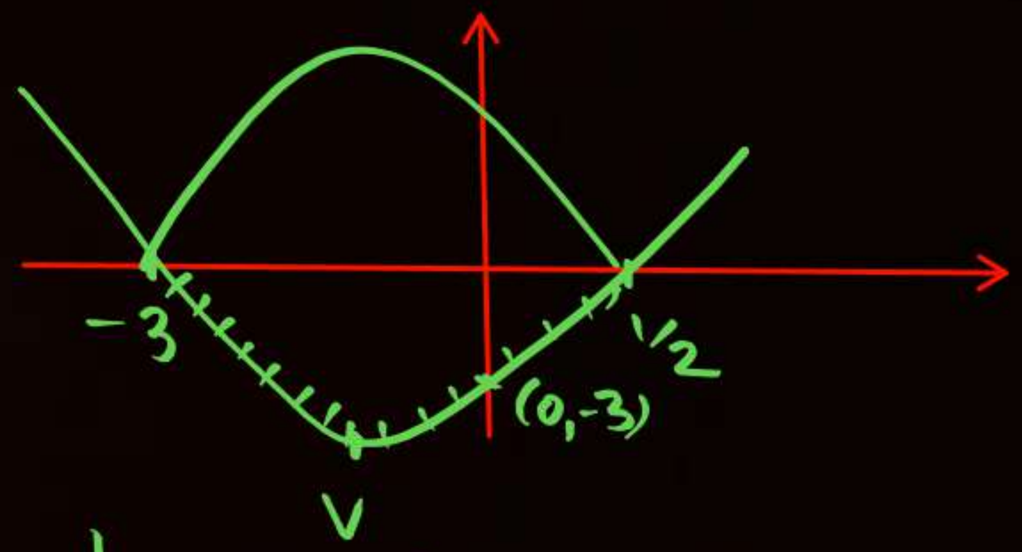
(C) 5

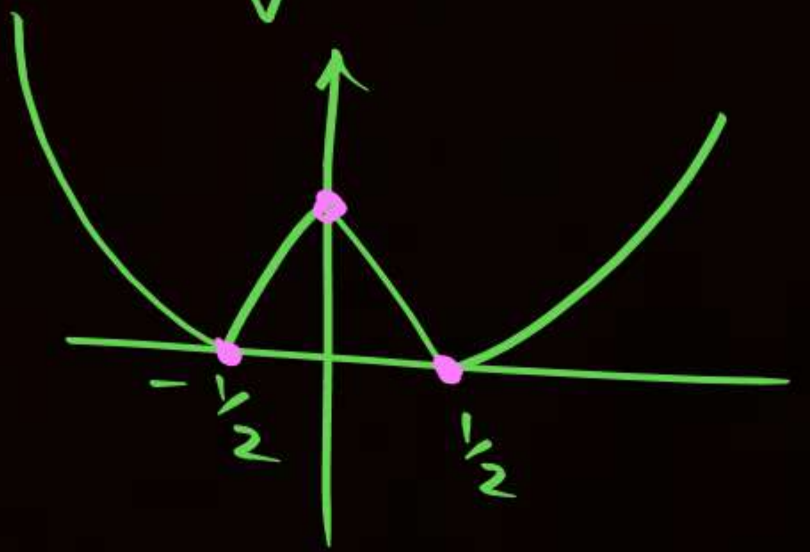
$\boxed{f(x) = g(|x|)}$

(D) 3 ✓

$\boxed{y = 2x^2 + 5x - 3}$

$y = 2x^2 + 6x - x - 3$
$y = (2x-1)(x+3)$



QUESTION [JEE Main 2024 (Jan. I)]

Let $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3), x \in R$. Then $f'(10)$ is equal to ______

$a \quad b \quad c$

$$\boxed{f(x) = x^3 + ax^2 + bx + c}$$

$$\boxed{f'(x) = 3x^2 + 2ax + b} \Rightarrow \text{Put } x = 1 \Rightarrow \boxed{3 + 2a + b = a}$$

$$f''(x) = 6x + 2a \longrightarrow \text{Put } x = 2 \quad \boxed{12 + 2a = b} \quad \boxed{a + b = -3}$$

$$f'''(x) = 6 \quad \text{Put } x = 3 \quad f'''(3) = \boxed{6 = c}$$

$$12 + 2a = -3 - a$$

$$3a = -15$$

$$\boxed{a = -5}$$

$$-5 + b = -3$$

$$b = 2$$

$$\boxed{f(x) = x^3 - 5x^2 + 2x + 6}$$

## QUESTION [JEE Main 2024 (Jan. II)]

$f'(x) = \frac{1}{2} - \frac{2}{x^2}$

$\frac{x^2 - 4}{2x^2}$

Consider the function $f : (0, 2) \to \mathbf{R}$ defined by $\boxed{f(x) = \frac{x}{2} + \frac{2}{x}}$ and the function $g(x)$ defined by $g(x) = \begin{cases} \min\{f(t)\}, & \boxed{0 < t \le x} \text{ and } 0 < x \le 1 \\ \frac{3}{2} + x, & 1 < x < 2 \end{cases}$. Then,

$f(x)$

$\frac{(x-2)(x+2)}{2x^2} < 0$ in $(0,2)$

-ve, +, +

$f(x)$ is $\downarrow$

A) g is not continuous for all $x \in (0, 2)$

B) g is continuous and differentiable for all $x \in (0, 2)$

C) g is continuous but not differentiable at $x = 1$ ✓

D) g is neither continuous nor differentiable at $x = 1$

$g(x) = \begin{cases} \frac{x}{2} + \frac{2}{x}, & 0 < x \le 1 \\ \frac{3}{2} + x, & 1 < x < 2 \end{cases}$

$\left.\begin{array}{l} LHL = \frac{1}{2} + \frac{2}{1} = \frac{5}{2} \\ RHL = \frac{3}{2} + 1 = \frac{5}{2} \end{array}\right] \Rightarrow$ cont at $x = 1$

LHD =

$\frac{1}{2} - \frac{2}{x^2} \quad x = 1$

$\frac{1}{2} - \frac{2}{1} = -3/2$

RHD $\rightarrow$ 1

non diff.

## QUESTION [JEE Main 2024 (Jan. I)]

Let $g(x)$ be a linear function and $f(x) = \begin{cases} g(x), & \boxed{x \le 0} \\ \left(\frac{1+x}{2+x}\right)^{\frac{1}{x}}, & x > 0 \end{cases}$, is continuous at $x = 0$.

If $\boxed{f'(1) = f(-1)}$, then the value $g(3)$ is

(A) $\frac{1}{3}\log_e\left(\frac{4}{9}\right) + 1$

(B) $\log_e\left(\frac{4}{9e^{1/3}}\right)$

(C) $\frac{1}{3}\log_e\left(\frac{4}{9e^{1/3}}\right)$

(D) $\log_e\left(\frac{4}{9}\right) - 1$

$g(x) \to ax + b$

$LHL = b$

$RHL = \left(\frac{1}{2}\right)^{\infty} \to 0$

$\boxed{b = 0}$ ✓

$f(-1) = a(-1) = -a$

$y = \left(\frac{1+x}{2+x}\right)^{1/x} \Rightarrow x = 1 \Rightarrow y = 2/3$

$\ln y = \frac{1}{x} \ln\left(\frac{1+x}{2+x}\right)$

$\ln y = \frac{1}{x}\left(\ln(1+x) - \ln(2+x)\right)$

$x \ln y = \ln(1+x) - \ln(2+x)$

$\ln y + \frac{x}{y} y' = \frac{1}{1+x} - \frac{1}{2+x}$

$g(x) = ax$

$\boxed{g(3) = 3a} = ?$

$$\ln(2/3) + \tfrac{3}{2}y' = \tfrac{1}{2} - \tfrac{1}{3}$$

$$\ln(2/3) + \tfrac{3}{2}y' = \tfrac{1}{6}$$

$$\tfrac{3}{2}y' = \tfrac{1}{6} - \ln(2/3)$$

$$\tfrac{3}{2}y' = \tfrac{1}{6} + \ln(3/2)$$

$$y'(1) = \tfrac{2}{3}\left[\tfrac{1}{6} + \ln 3/2\right] = -a$$

$$-3a = 2\left[\tfrac{1}{6} + \ln 3/2\right]$$

$$-3a = \tfrac{1}{3} + 2\ln(3/2)$$

$$3a = -\tfrac{1}{3} - 2\ln(3/2)$$

$$\boxed{3a = -\tfrac{1}{3} + \ln(4/9)}$$

$$= -\tfrac{1}{3}\ln e + \ln(4/9)$$

$$= -\ln e^{1/3} + \ln 4/9$$

$$= \ln\left(\frac{4}{9e^{1/3}}\right)$$

QUESTION [JEE Main 2024 (Feb. II)]

$3\cancel{96}\times\frac{35}{\cancel{32}} = 35\times 3$

$= 105$

If $y = \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}} + \frac{1}{15}(3\cos^2 x - 5)\cos^3 x$, then $96y'\left(\frac{\pi}{6}\right)$ is equal to :

$$\frac{(\sqrt{x}+1)\cancel{\sqrt{x}}\left[\underbrace{x^{3/2}-1^{3/2}}\right]}{\cancel{\sqrt{x}}\left[x+\sqrt{x}+1\right]}$$

$$\frac{(\sqrt{x}+1)\,(x^{1/2}-1)\,\cancel{\left(x+1+x^{1/2}\right]}}{\cancel{(x+\sqrt{x}+1)}}$$

$(\sqrt{x}+1)(\sqrt{x}-1)$

$(x-1)$

$y = x-1+\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x$

$y' = 1+\cos^4 x(-\sin x)-\cos^2 x(-\sin x)$

$y' = 1+\left(\sqrt{3}/2\right)^4\left(-\frac{1}{2}\right)+\left(\sqrt{3}/2\right)^2\cdot\frac{1}{2}$

$y' = 1+\frac{9}{16}\left(-\frac{1}{2}\right)+\frac{3}{8}$

$1-\frac{9}{32}+\frac{3}{8}$

$y' = \frac{32-9+12}{32}$

$y' = 35/32$

## Homework

DPP

PYQs → 2019-20
21


Thank You
# IIT Phodna Hai !